Essays on Real Exchange Rates and Theoretical Monetary Aggregation

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Abstract

This dissertation is a collection of three essays focused on real exchange rates and theoretical monetary aggregation. The first essay focused on the convergence of real exchange rates' idiosyncratic effects after isolating their common feature, which was caused by binary exchange rate numeraire. The evidence of convergence of idiosyncratic real exchange rates was mixed. The half-life of real exchange rates mean reverting period was considerably shorter than what was previously found in literature. The result was significantly shorter than "Rogoff's consensus" half-life which was 3- to 5-year.

The second essay was to construct Divisia index for China following Barnett (1978, 1980). First, we probed the statistical discrepancy of the raw data provided by the People's Bank of China and proposed the appropriate forms that the authority has been published to extract the data of monetary assets' balances. Second, we adjusted the interest rates of China monetary assets to the annualized one-month holding period yields and further we used the yield curve adjustment method to subtract the term premium of monetary assets with different maturities. Third, we constructed the nominal Divisia index of Renminbi M2. The constructed Divisia M2 is seasonally unadjusted.

In the third essay, the theoretical monetary aggregation was introduced to a new Keynesian Dynamic Stochastic General Equilibrium framework. It has been argued by Barnett and Chauvet (2011a) and Barnett (2011) that a major source of inaccurate information within agents' information sets was the troublesome monetary aggregate data the Fed provided. Those data were inconsistent with elementary principles of aggregation theory over imperfect substitutes. By introducing Divisia monetary aggregation into a New Keynesian DSGE model, we showed that the prevailing simple-sum monetary aggregates violated decision optimality conditions and thereby distorted decisions. With a continuum of monetary assets and a monetary aggregate, we developed an internally coherent money-demand function, improving the understanding of the demand for "moneyness."The user-cost aggregate, which was dual to the monetary aggregate and was interpreted as the price of "moneyness,"played the key role in our framework and was preferable to the interest rate aggregate or single interest rate most commonly used within such models. We proposed a monetary policy rule consistent with the model and then studied the impulse responses.

Keyword: dynamic factor model, common feature estimation, cross sectional dependence, idiosyncratic real exchange rates dynamics, general-to-specific method, panel data model, panel unit root test, China, monetary services, money demand, Divisia index, theoretical monetary aggregation, measurement error, aggregation, index number theory, Central Bank of China, Federal Reserve, DSGE model.

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Chapter 1

Introduction

My dissertation is a collection of three essays which focus on international economics and theoretical monetary aggregation. The rest of the dissertation is organized as following: section 2 will introduce my first essay, The Idiosyncratic Dynamics of the Real Exchange Rates. I use a dynamic factor model to isolate the real exchange rates idiosyncratic dynamics from the contamination of the numeraire country's unexpected shock. I then propose a new estimation of the common feature to capture the numeraire shock. The convergence of idiosyncratic dynamics of real exchange rates is tested and converging speed is estimated by adopting a Two Stage Least Square method. Besides parameter identifying within the test and the estimation, we perform data-based experiments by varying *ad-hoc* parameters. We find that under the common feature, the evidence of real exchange rates convergence for 20 OECD countries from 1973 to 1998 is much weaker than previous literature documented. And also, the convergence of real exchange rates is not robust to priori parameter settings of lag length selection combining with size control of serial correlation. However, the evidence of idiosyncratic real exchange rates convergence is stronger in the 1990's. For some cases of priori parameter specifications, the point estimation of half-life could be 18 months with a very tight confidence interval below three years.

Section 3 will show my second essay, which reviews Barnett (1980)'s Divisia monetary index theory, discusses the adjustment of interest rates for China's own case and constructs the Divisia index of Chinese Renminbi. This section is devoted into constructing of the accurate monetary aggregation of China's RMB. Yu and Tsui (2000) was the first attempt to construct China's monetary aggregation using Divisia index, for which they referred as Monetary Service Index (MSI) following Anderson et al., (1997a, 1997b). Their data set consisted of 168 monthly observations from January 1984 to December 1997, which is extracted from Monthly Statistics of China published by the State Commission of Statistics of China. However, the rapid economy growth and quick financial innovations implies potential structural changes of China's monetary demand, thus the up-to-date version of Divisia index of RMB is needed for further study purpose.

Section 4, which is my third essay, shows a New Keynesian framework including theoretical monetary aggregation in household's utility function. It has been argued by Barnett and Chauvet (2011a) and Barnett (2011) that a major source of inaccurate information within agents' information sets was the troublesome monetary aggregate data the Fed provided. Those data are inconsistent with elementary principles of aggregation theory over imperfect substitutes. By introducing Divisia monetary aggregation into a New Keynesian DSGE model, we show that the prevailing simple-sum monetary aggregates violate decision optimality conditions and thereby distort decisions. With a continuum of monetary assets and a monetary aggregate, we developed an internally coherent money-demand function, improving understanding of the demand for "moneyness". The user-cost aggregate, which is dual to the monetary aggregate and is interpreted as the price of "moneyness", plays the key role in our framework and is preferable to the interest rate aggregate or single interest rate most commonly used within such models. We propose a monetary policy rule consistent with the model, and study the impulse response to monetary policy shock. Section 5 concludes.

Supplementary estimation algorithm, and DSGE framework development are provided in appendix.

Chapter 2

The idiosyncratic dynamics of the real exchange rates

The state of the art of literature on real exchange rates (RER) convergence is not conclusive when we account for cross-sectional dependence across countries. Providing evidence of RER convergence under cross-sectional dependence is one aim of this essay. But in this essay we do not directly test the Purchasing Power Parity (PPP), rather we study the potential resource of non-stationarity which pretends the convergence of real exchange rates.

2.1 Introduction

Recent decades witness a large growing literature of dynamic panel testing of Purchasing Power Parity (PPP). See Frankel and Rose (1996), Oh (1996), Wu (1996), Bayoumi and MacDonald (1999), Papell and Theodoridis (2001), Taylor (2001), Papell (2002), Imbs et al. (2005), Papell and Prodan (2006), and Robertson et al. (2009) among others. However, most of literature suppose cross-sectional independence of the real exchange rates, which is quite a simplified assumption. In fact, one can easily observe co-movements of real exchange rates when the numeraire country experiences an unexpected shock. In contrast, O'Connell (1998), Taylor et al (2001), Murray and Papell (2002), Murray and Papell (2005), and Choi et al. (2006) adopted non-linear model or General Least Square (GLS) technique to account for potential cross-sectional dependence. The conclusion of literature mainly involves two parts. First, unit root hypothesis of PPP might be rejected in panel data structure under cross-sectional independence. This is partially because that panel unit root test has more power than univariate time series unit root test, which is documented in Maddala and Wu (1999), Levin et al. (2002) and Im et al. (2003) among others. Second, either using univariate time series or panel data framework, the estimated PPP mean reverting period, if it is not infinite, is much longer than nominal price rigidity period. Such inconsistency of estimated PPP mean reverting period and nominal price stickiness period is concluded as the "PPP puzzle", see Rogoff (1996).[1]

Along with evidence supporting PPP, many criticizes rise around testing technique and data transformation. Banerjee et al. (2005) argued that we should avoid "automatically" adopting panel structure to estimate PPP because of cross-sectional dependence. Econometric theory literature also found that with presence of cross-sectional dependence but assuming cross-sectional independence, the size of "first generation" panel unit root test increases dramatically.[2] In Banerjee's words, "the unit root null is rejected too often". When allowing for cross-sectional dependence, Chang (2002), Choi (2002), Moon and Perron (2004), Bai and Ng (2004), Breitung and Das (2005), Phillips and Sul (2007) and Pesaran (2007) among others developed so-called "second generation" panel unit root test. Chang (2002) proposed an instrument method to adjust the cross-section dependence. Phillips and Sul (2003), and Moon and Perron (2004) adopted a common factor structure in the error component and independently developed an Orthogonal Procedure (OP) to accommodate first generation unit root test with cross-section dependence. Bai and Ng (2004) used a more general framework and introduced PANIC (Panel Analysis of Nonstationarity in Idiosyncratic and Common

[1]However, Imbs et al. (2005) tried to adjust cross-sector-aggregation bias to conclude a shorter mean reverting period. Chen and Engel (2005) pointed out the statistical insignificance of aggregation bias. Choi et al. (2006) considered three potential biases of PPP estimation and concluded that the puzzle still holds.

[2]The first generation of panel unit root tests include but not limited to Maddala and Wu (1999), Hadri and Phillips (1999), Choi (2001), Levin et al. (2002), Im et al. (2003), and Shin and Snell (2006). Baltagi and Kao (2000) provided an early review.

components).

Such new panel unit root tests provided us a fresh outlook of PPP hypothesis. Recently, Smith et al. (2004), Banerjee et al. (2005), Pesaran (2007), Mark and Sul (2008), and DeSilva el al. (2009) explicitly allow cross-sectional dependence to test PPP. Using the same quarterly dataset, Smith et al. (2004), and Pesaran (2007) found that for some particular test statistics we can reject the unit root hypothesis when including 2 quarters lags or more in paneled $AR(p)$ process. In contrast, Mark and Sul (2008) used log-t test to probe a disaggregated dataset, which is also used by Imbs et al. (2005), and found no solid evidence of Law of One Price (LOP) at all.

Therefore, when we account cross-sectional dependence across countries, the evidence of real exchange rates' convergence is weak. Providing such evidence is one aim of this essay. But in this essay we do not test for PPP, rather we study the potential resource of nonstationarity which pretends the convergence of real exchange rates. The cross-sectional dependence is formulated as a common feature in our framework. We first study the common feature of real exchange rates which is caused by adopting a single numeraire. When the numeraire country experiences an unexpected shock, all other countries' real exchange rates were affected by the shock and might exhibit nonstationarity. We use a general single factor model to isolate the idiosyncratic real exchange rates dynamics from the contamination of the numeraire country's shock. In the model, the common factor accounts for the numeraire country's effect and the idiosyncratic components account for other countries' "de-featured" real exchange rates dynamics. We propose a new common feature estimation and also estimate the common feature using the principal component method as a benchmark. Then the idiosyncratic components are consistently estimated without any assumption of stationarity. Either numeraire country's real exchange rate or other countries' real exchange rates or both are allowed to be $I(0)$ or $I(1)$. After isolating the idiosyncratic effects from the common feature, the convergence of real exchange rates is tested and converging speed is estimated by adopting a Two Stage Least Square method.[3] Besides parameter identifying within the test and the estimation, we perform data-based experiments by varying exogenous parameter specifications. The experiments show that under the common feature, there is evidence of convergence of real exchange rates from 1973 to 1998, though the evidence is much weaker than previous literature documented for some priori parameter settings. And also, the convergence of real exchange rates is not robust to the lag length selection combining with size control of serial correlation. Our tests tend to reject the nonstationary null either with more lags or with "looser" control of serial correlation. However, the evidence of idiosyncratic real exchange rates convergence is stronger in the 1990's. For some cases of priori parameter specifications, the point estimation of half-life could be 18 months with a very tight confidence interval below three years.

The contribution of this essay can be concluded in at least three aspects. First, we isolate countries' idiosyncratic real exchange rates dynamics from the common effect of cross-country co-movement. The impact of common feature on individual countries' real exchange rates (the factor loadings) and idiosyncratic components (the residuals) are consistently estimated. Second, we propose to use the real effective exchange rate (REER) of the numeraire country as a new candidate of the common feature estimation. Using time-varying trading volumes as weights to compute the average, REER provides comprehensive information of the common feature than its ancestors considered in literature. Third, when accounted for the common feature, the evidence of convergence is interestingly mixed from our result. The results also imply structural breaks of real exchange rates before 1988. When we exclude the great appreciation and depreciation of the US dollars in the middle of 1980's, the evidence of convergence is stronger.

[3]The estimation method is probably first proposed by Levin et al. (2002). We will show further that empirically this method tends to choose less lags and obtains more statistically efficient inferences. Banerjee et al. (2005) also confirmed that even assuming cross-section independence, Levin's test suffers least from the size distortion among the first generation panel unit root tests.

2.2 A general single factor model of the real exchange rates under cross-section dependence

Let s_{it} be the logarithm of direct quoted nominal exchange rate of country i. p_{0t} is the logarithm of the numeraire country's price level and p_{it} is the logarithm of country i's price level. $i = 1, ..., N$ is the cross-section individual countries we are interesting to study, $t = 1, ..., T$ stands for the time periods. The real exchange rate of an arbitrary country i can be obtained by

$$q_{it}^{(0)} = s_{it} - p_{it} + p_{0t}, \tag{2.1}$$

where the superscript 0 in parentheses of q means the country i's real exchange rate is anchored on numeraire country 0. Meanwhile, the real relative price of the numeraire country considering country i as numeraire can be immediately obtained by $q_{0t}^{(i)} = -q_{it}^{(0)}$, for $i = 1, ..., N$.

When estimating dynamic panel of real exchange rates, the cross-sectional dependence is usually ignored.[4] However, from equation (2.1) one can easily find a nonzero covariance of real exchange rates between country i and j. This is partially because the shock of price level of numeraire country spreads out to all other countries' real exchange rates. To illustrate the idea, assuming mutually independence of nominal exchange rates s_{it} and price levels p_{it} at this moment only, then one could obtain $E\left[q_{it}^{(0)} q_{jt}^{(0)}\right] = E\left[p_{0t}^2\right]$ which is generally nonzero.

To account for the cross-sectional dependence of panel data estimation, Phillips and Sul (2003), Moon and Perron (2004) independently developed econometric theory which adopted factor-structured models and included the common features in the error component of the dynamic panel model to capture the cross-sectional correlation. Their methodology treats the common feature as well as the factor loadings as nuisance parameters and then develops consistent orthogonal procedure (OP) to "de-factor" the original data. The basic idea of those studies is to project the data on the space of estimated factor loadings to yield

[4] See Mark and Sul (2008) for a brief review of such pitfalls in literature.

cross-sectional independent data, and then apply typical dynamic panel data estimating and testing technique. Their proposed panel unit test has good asymptotic power properties when the model does not contain deterministic (incidental) trends, which is suitable for panel real exchange rates estimating and testing.

However, we are not only interested in testing for unit roots of real exchange rates, but also in the nature of the cross-sectional correlation of the real exchange rates as well as the resources of nonstationarity of the real relative prices usually found in empirical literature. Therefore we use a general factor model which casts the common feature in the regressors instead of in the error components. A similar model has been used by Stock and Watson (1998), Bai and Ng (2002), Bai and Ng (2004), Bai (2003), and Mark and Sul (2008). Bai and Ng (2004) developed PANIC (Panel Analysis of Nonstationary in Idiosyncratic and Common components) which is the main building block of our model setup.

Let $q_{it}^{(0)}, i = 1, ..., N, t = 1, ...T$ to be the real exchange rate of country i at period t considering the country 0 as numeraire is generated by

$$q_{it}^{(0)} = c_i + \lambda_i F_t + e_{it} \tag{2.2}$$

$$(1 - \rho_0 L) F_t = C(L) u_t \tag{2.3}$$

$$(1 - \rho_i L) e_{it} = D_i(L) \varepsilon_{it} \tag{2.4}$$

where $C(L) = \sum_{j=0}^{\infty} C_j L^j$ and $D_i(L) = \sum_{j=0}^{\infty} D_{ij} L^j$ with L denotes the lag operator. c_i is the individual-specific effect with $c_i \sim iid(0, \sigma_c^2)$. We assume (i) $u_t \sim iid(0, \sigma_u^2)$ and for each i, $\varepsilon_{it} \sim iid(0, \sigma_{\varepsilon i}^2)$; (ii) $E(\varepsilon_{it}\varepsilon_{jt}) = 0$ for any $j \neq i$; (iii) the error terms u_t , ε_{it} and the factor loading λ_i are mutually independent.[5]

By construction, the idiosyncratic error component e_{it} is orthogonal to the common

[5] Assumption (i) and (iii) are in line with Phillips and Sul (2003), Moon and Perron (2004), and Bai and Ng (2004) and (ii) is explicitly considered as a basic assumption of Mark and Sul (2008). In Bai and Ng (2004), a parallel assumption is $E(\varepsilon_{it}\varepsilon_{jt}) = \tau_{ij}$ with $\sum_{i=1}^{N} |\tau_{ij}| \leq M$ for all j. This assumption allows weak cross-section independence and then their model is an "approximate factor model" in the sense of Chamberlain and Rothschild (1983). But clearly our assumption is a special case of Bai and Ng (2004).

factor F_t. We are interested to probe the resources of possible nonstationarity of $q_{it}^{(0)}$. It is important to analysis the "de-featured" e_{it}, since the consistent estimation is our priority. If F_t is $I(0)$ and e_{it} is $I(1)$, then the nonstationarity is from individual countries' own nature of persistence. If F_t is $I(1)$ and e_{it} is $I(0)$, then the common feature across the countries accounts for the violation of the purchasing power parity as a valid condition in the long-run. Notice that when both F_t and e_{it} are $I(1)$, directly regress $q_{it}^{(0)}$ on F_t is spurious (Bai and Ng, 2004) thus the estimation of factor loadings λ_i and residuals e_{it} will be inconsistent. In this case, however, the first difference of common factor and idiosyncratic terms are stationary, so that we can consistently estimate λ_i by the first difference of equation (2.2). And also, even if either F_t or e_{it}, or both, are $I(0)$, over difference the data still yield consistent estimate of λ_i, thus consistent estimate of idiosyncratic components e_{it}. To fix the idea, denote $x_{it} = \Delta q_{it}^{(0)}$, $f_t = \Delta F_t$, $z_{it} = \Delta e_{it}$ to be the change of real exchange rates, the change of common feature and the change of idiosyncratic dynamics of real exchange rates, respectively. Then the model (2.2) in first-order difference form can be rewritten as

$$x_{it} = \lambda_i f_t + z_{it}, \tag{2.5}$$

where $\{x_{it}\}$ is a set of $(T-1) \times N$ stationary variables. If we assume the common feature F_t is unknown then we estimate it using principal component method. Otherwise we will use some observable variables to approximate F_t . We will discuss the estimations of F_t in the next section in detail. Denote the estimated common feature by $\hat{F}_t$ and its first-order differenced estimation by $\hat{f}_t$.

Given the first-order differenced estimation of common feature $\hat{f}_t$, we estimate the factor loadings λ_i as well as the idiosyncratic component z_{it} by ordinary least square. Denote those estimations by $\hat{\lambda}_i$ and $\hat{z}_{it}$. Accumulate $\hat{f}_t$ and $\hat{z}_{it}$ through $t = 2, ..., T$ yields

$$\hat{F}_t = \sum\nolimits_{s=2}^{t} \hat{f}_s \tag{2.6}$$

$$\hat{e}_{it} = \sum\nolimits_{s=2}^{t} \hat{z}_{is}, i = 1, ..., N. \tag{2.7}$$

Under the model setup, the cross-section correlation of real exchange rates' depreciation rates is determined by the common feature with extension of the factor loadings, i.e., $E\left(\Delta q_{it}^{(0)} \Delta q_{jt}^{(0)}\right) = \lambda_i E\left(f_t f_t^{\prime}\right) \lambda_j$.

Before moving on to the common feature estimation, we need to make a few comments to clarify our model. First, we assume a single factor structure in equation (2.2) and the factor loading λ_i is a scalar. This is not only for simplicity but also with economic reasoning. Because the correlation of panel real exchange rates is generated by the numeraire country, shocks oriented from the numeraire country will spread out through channels to the rest of the world. The single common factor F_t calibrates the dynamic behavior of the numeraire country's exchange rate. Second, both the common factor and the idiosyncratic error are assumed to be $ARIMA$ process in our model setup. Empirical estimates will adopt pure autoregressive structure which allows serial correlation. The common factor process is nonstationary if ρ_0 in (2.3) is unity and idiosyncratic components are nonstationary if ρ_i in (2.4) are unity for some country i. Notice that we allow each idiosyncratic component process to follow different dynamics by allowing its serial correlation $D_i(L)$ to be different across countries.

2.3 Estimating the common feature

In this section we focus on the common feature that generates the cross-sectional correlation of different country's real exchange rates. Throughout the literature, the common feature is defined as a feature of a group of series if there exists a nonzero linear combination of these series that does not have the feature. The linear combination coefficient vector is called cofeature vector. Urga (2007) provided a good theoretical review and examples of common features in application. In this essay we compare three alternative estimations of the common factor. The first is simple (arithmetic) average of real exchange rates which is commonly used in empirical literature. For the second we propose to use real effective exchange rate (REER) of the numeraire country as an approximation of the common factor. And the third estimation is assuming that the common factor is unobserved but can be estimated by Principal Component method.[6] Denote three alternative estimations as F_t^{A1}, F_t^{A2} and F_t^{A3}, respectively.

Alternative I. Imbs et al. (2005) and Mark and Sul (2008) used arithmetic average of contemporaneous real exchange rates as an approximation of the unobserved common feature. Pesaran (2006) proposed to filter the individual-specific regressors by the means of (weighted) cross-section aggregates such that the effect of unobserved common factors is asymptotically eliminated as the cross-section dimension tends to infinity.[7] In a companion paper, Pesaran (2007) proposed to proxy the common factor by the cross-section (arithmetic) mean of the data and the lagged values of the mean for N sufficiently large. Thus, to be in line with literature, cross-section (arithmetic) average is our *alternative I* (denote $A1$). To be specific,

$$F_t^{A1} = \frac{1}{N} \sum\nolimits_{i=1}^{N} q_{it}^{(0)}. \tag{2.8}$$

Notice that obviously $\Delta \hat{F}_t^{A1}$ treats all other countries' real exchange rate $q_{it}^{(0)}$ indifferently. The hidden assumption behind $A1$ is that all countries are homogeneous, which seems to simplified in practice. Fisher (1922) illustrated the disadvantage of arithmetic average in the index theory. This is simply because equal-weighted average explicitly assumes one-to-one substitution of components in the group of series. Andrews (2005) argued that the impact of common features generally is not the same across different units. However, as long as literature argued, This measurement is asymptotically accurate as the cross-sectional dimension increases. Denote $\hat{f}_t^{A1} = \Delta \hat{F}_t^{A1}$.

[6]See Eickmeier and Breitung (2009) for a brief introduction of the estimation technique of principal component method.

[7]See Pesaran (2006) for his method of choosing appropriate weights.

Alternative II. Empirically, countries are heterogeneous and if they are not, there is no need to adopt panel data model at all: time series of an arbitrary country is adequate under assumption of homogeneity. In the spirit of heterogeneity, we propose to proxy the common factor using the real effective exchange rate (REER) of the numeraire country. The merit of REER is that it is observable, already calculated by authorities and ready to use. Since two countries' relative prices are bilateral, the impacts of different countries on the numeraire country are also different. For example, if we consider the U.S. as numeraire, Japanese real exchange rate has much stronger impact on the U.S. REER than Denmark or Norway does. But if we consider Germany as the numeraire, the impact order could be reversed.

Keeping this reasoning in mind, we should use some kind of different weighted average to estimate the common feature, so that the weights could reflect the strength of impact of different countries on the numeraire country. Time varying trading weights is a good candidate. It has a long tradition of the importance of the trading good in the international economics studies. In our real exchange rates context, one can obtain an approximation[8] of the real effective exchange rate of the numeraire country as $\bar{q}_{0t}^{(ROW)} = \sum_{i=1}^{N} \omega_{it} q_{0t}^{(i)}$ where $\sum_{i=1}^{N} \omega_{it} = 1$ is the trading weight and $\bar{q}_{0t}^{(ROW)}$ is the real relative price of the numeraire country considering the rest of world (ROW) as numeraire. Our second alternative of the common factor (denoted $A2$) can be obtained by

$$F_t^{A2} \equiv \sum_{i=1}^{N} \omega_{it} q_{it}^{(0)}. \tag{2.9}$$

We need to further clarify that the common feature is equivalent to but not identical to the REER of the numeraire country. The common feature measures the impact of shocks of the numeraire country on the rest of the world, whereas the REER measures the impact of

[8]By approximate we mean the "major effective exchange rate" defined by the Federal Reserve Board. Across years, especially the recent decade, the trading weights of developing countries count more and more in the US trading partners (approximately 50%). However, most developing countries choose the fixed exchange rate regime, thus excluding those countries will not significantly bias our estimation.

shocks of the rest of the world on the numeraire country. The common feature itself should be based on the numeraire country. Therefore we need to convert $\bar{q}_{0t}^{(ROW)}$ to be country 0 currency based. Notice that real relative price of country 0 using country i as numeraire is only the opposite of real relative price of country i with respect to country 0 as numeraire, one can rewrite equation (2.9) as

$$F_t^{A2} \equiv \sum\nolimits_{i=1}^{N} \left(\omega_{it} q_{it}^{(0)}\right) = \sum\nolimits_{i=1}^{N} \left(-\omega_{it} q_{0t}^{(i)}\right) = -\bar{q}_{0t}^{(ROW)}.$$

In other words, the common factor is the opposite of numeraire country's real effective exchange rate. Denote $\hat{f}_t^{A2} = \Delta \hat{F}_t^{A2}$.

Alternative III. Our third alternative calibration of common factor (denoted $A3$) is estimated by the Principal Component method. Choi (2002), Moon and Perron (2004), Bai and Ng (2002), Bai and Ng (2004), and Bai (2003) adopted this method to estimate the unobserved common factor. The principal component estimator of first-order difference common factor is denoted $\hat{f}_t^{A3}$, which is the eigenvector corresponding to the largest eigenvalue of the data matrix $x_{it}x_{it}^{'}$. In addition, We found that the largest eigenvalue accounts for 62.56% of all eigenvalues. One is aware that the eigenvector is not unique unless we restrict its norm. To make our three alternative common feature comparable under the same scale, we normalize $\hat{f}_t^{A3}$ by the average of norms of $\hat{f}_t^{A1}$ and $\hat{f}_t^{A2}$, i.e.,

$$\left\| \hat{f}_t^{A3} \right\| = \frac{1}{2} \left(\left\| \hat{f}_t^{A1} \right\| + \left\| \hat{f}_t^{A2} \right\| \right)$$

where $\|A\| = trace\left(A^{'}A\right)^{1/2}$ is the Euclidian norm.

Throughout this essay we adopt the United States as the numeraire country. Figure 2.1 illustrates the long-run dynamic behavior of three alternatives of the common features. As we can see from Figure 2.1, all the three captured the great depreciation and appreciation of the US dollar, which is defined by using US REER. This is because the alternative $A1$

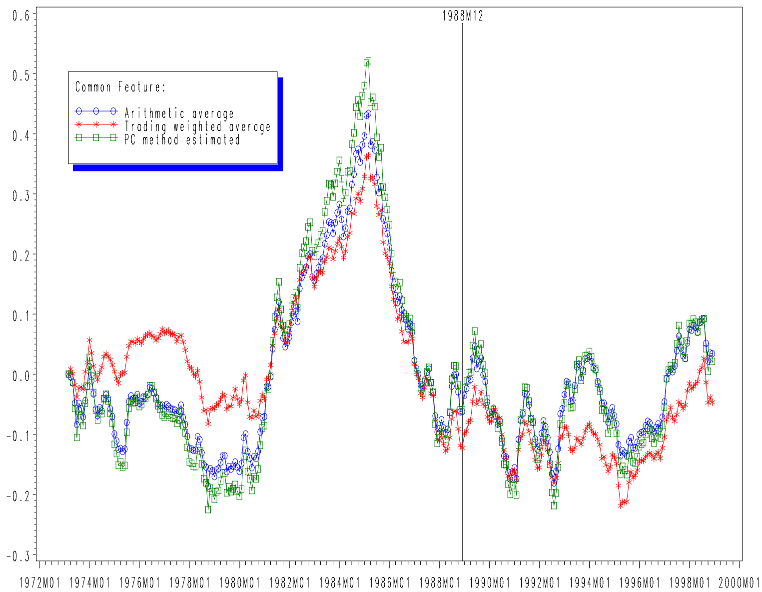

Figure 2.1: Estimated Common Feature

and $A3$ can be viewed as approximations to the US dollar real effective exchange rate in some degree. The US dollar appreciated from 1981 to 1985 and depreciated from 1985 to 1987. Papell (2002) studied effects of this great deprecation and appreciation period as a source of violation of PPP. Notice that the arithmetic average $A1$ and the principal component estimated $A3$ overlapped each other when the US dollar real exchange rates are relatively stable. However, $A1$ tends to be less volatile than $A3$ during most of the time. In contrast, interestingly, the trading weighted average $A2$ significantly departs from $A1$ and $A3$ from 1973 to 1981 and from 1989 to 1999, and tends to be more close to zero before 1982. However, the relative low volatility of $A2$ broke down during and after the great appreciation and depreciation of US dollar.

Now we turn to the first difference of the three alternatives in Figure 2.2. The change of the common feature proxies the depreciation rate of the US dollar. We truncated the data to exclude the great depreciation and appreciation period of US dollar and reports its real depreciation rate in a sample period from December 1988 to December 1998. All three candidates' depreciation rates exhibit very close to each other. However, $A3$ has the most volatility and $A2$ seems to be the least volatile.

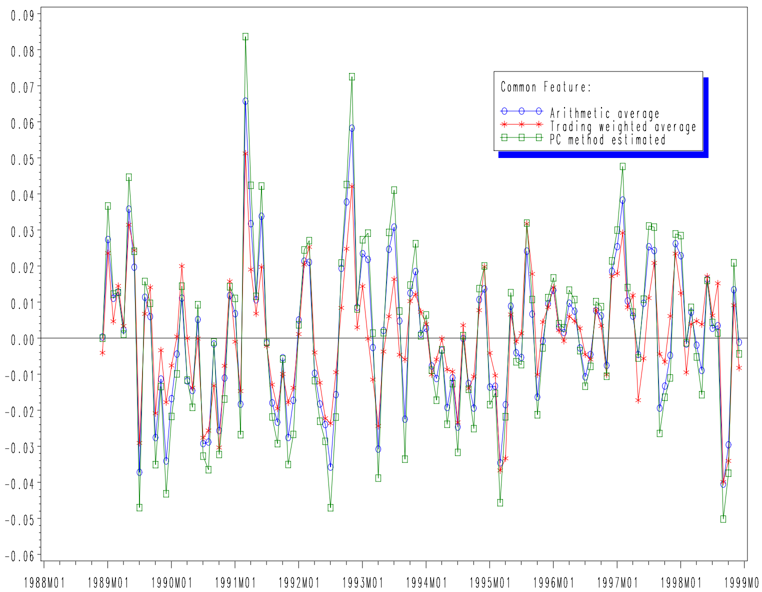

Figure 2.2: Estimated Depreciation Rate of the US Dollar

We have also tested the stationarity of three alternative estimations of the common factor using augmented Dickey-Fuller (ADF) test and Pillips-Perron (PP) test. We found that all the three estimations of common feature significantly fail to reject the null of nonstationarity in level but significantly reject the null in first difference, in 5% significance. In other words, the common factor process is $I(1)$. Our tests further confirm the finding from Mark and Sul (2008). The test result is skipped to keep the context short and available upon request.

2.4 Idiosyncratic dynamics of real exchange rates

2.4.1 2SLS estimation of dynamic panel data

The estimated residuals $\hat{e}_{it}$ from (2.5) and (2.7) are purely the idiosyncratic effects of countries' real exchange rates after isolating the common feature. To empirically estimate the theoretical model (2.4), we convert its $ARIMA$ structure to be reduced $AR(p)$ structure. Suppose we are interested to control the serial correlation of errors to be in the size α, the estimated panel regression model is

$$\Delta \hat{e}_{it} = d_i + \delta \hat{e}_{it-1} + \sum_{L=1}^{P_i} \theta_{iL} \Delta \hat{e}_{it-L} + error_{it} \tag{2.10}$$

for $i = 1, ..., N$ and $t = 2, ...T$. We then apply a dynamic panel estimation method proposed by Levin et al. (2002) (thereafter "LLC"). Murray Papell (2005) adopted the same estimating technique as ours.[9] Although LLC's work was initially formulated to test panel data unit roots, it also provided an alternative approach to estimate dynamic panel data, especially suitable for the coefficients around unit root, which is the case for long-run convergence of real exchange rates. Since it is a two stage least square method, we denote this method as "LLC-2SLS". A brief revisit of the estimation technique is provided in appendix.

2.4.2 Lag length identification

When estimating dynamic panel data of autoregressive form, one of the basic issues is data-based lag-length identification. Hall (1994) proposed General-to-specific (GS) identification which is suggested by Harvey and Pierse (1984), and further advocated by Campbell and Perron (1991), Dickey (1994), Papell (2002) and Levin et al. (2002) among others. This essay adopts the GS instead of using AIC, SIC or HQIC. Given a sample length T, GS identification needs a priori maximum lag order $p_{\max}$, and then check the significance of θ_{iL} of equation (2.10) backwardly. The appropriate lag length p_i is determined when we first meet a significant coefficient at size α, which is also set in priori. It is obvious that we do not necessarily use $p_{\max}$ months as our estimating lag length. Thus after each appropriate lag order of univariate time series is determine, the average of those lag orders $\bar{p}$, where $\bar{p} = \frac{1}{N} \sum_{i=1}^{N} p_i$, is usually much smaller than $p_{\max}$.

The next concern is how long we should choose priori $p_{\max}$. A normal view of current-past dependence of real exchange rate is around two years, or 24 month, and some researchers

[9] However, their main focus is the mean unbiased estimator of dynamic panel $AR(p_i)$ in stead of direct estimation of the summation of the coefficients as we do here.

use no more than four quarters lags, see Pesaran (2007) for example. In this essay we set the largest $p_{\max}$ to be 24 months.

In next section we shall perform data-based experiments by varying the size of serial correlation α and maximum lag length $p_{\max}$. This is because the specification of lag length and significance to size control for serial correlation will typically change our empirical results, sometimes seriously distort them. If α is larger, it means that we are more "loosely" control for serial correlation in an $AR(p_{\max})$ structure. Notice that when we let $\alpha = 1$, we are actually forcing all time series to choose $p_{\max}$ lags. This case will also be reported in the next section as a benchmark.

2.5 Data experiments and empirical results

Our data is obtained from "International Macroeconomic Data Set" computed by the Economic Research Service (ERS) of the United States Department of Agriculture (USDA). The data set is published on the USDA website. It contains 79 countries with monthly real exchange rates form Jan. 1970 to Oct. 2008 and is calculated based on IMF International Financial Statistics (IFS) and Federal Reserve Board. We select 20 OECD countries as our computing subset. The countries are: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Greece, Ireland, Italy, Japan, Netherlands, New Zealand, Norway, Portugal, Spain, Sweden, Switzerland and the United Kingdom. The real exchange rates are US dollar based. At the beginning of January 1999, some of the European countries adopted Euro as a single currency, therefore we truncate the data to be ended at December 1998. Thus the total period is from March 1973 to December 1998, totally 310 months. In this essay we consider a total period 1973 - 1998 and subset period 1988 - 1998.

When we consider a long time span we should care about potential structural breaks. Vogelsang and Perron (1998) considered unit root testing and mean structural change of Purchasing Power Parity in time series case. Papell (2002) pointed out the great depreciation and appreciation of the US dollar in the 1980's could account for the weak rejection of panel unit roots in the US dollar-based real exchange rates for industrial countries. Smith et al. (2004) proposed to include a shorter period 1988-1998 which excludes the great depreciation and appreciation of the US dollar and less likely to be subject to structural breaks. We follow Smith et al. (2004) to study a sub-sample period, from January 1988 to December 1998, totally 120 months. The empirical results reported below are paralleled for the total sample period and the sub-sample period.

2.5.1 1973M03 - 1998M12

We perform a data-based parameter experiment of varying the maximum lag length $p_{\max}$ as well as varying the significance level α to control the size for serial correlation. We choose the significance level α to be 0.05, 0.25 and 1, and maximum lag length $p_{\max}$ to be 3, 6, 9, 12, 18, 24 months, respectively.[10] Table 2.1 reports panel unit root tests results using LLC test method after "de-factoring" the common feature. For each alternative estimation of common feature, namely $A1$, $A2$ and $A3$, the average lag length of all countries $\bar{p}$, the estimation of coefficient δ from the model (2.10) and corrected p-value are provided. The bottom block with $\alpha = 1$ is reported as a benchmark where we force all the individuals to select $p_{\max}$ lag length.

We found that evidence of convergence is mixing from Table 2.1. As we can see vertically from the table, typically $p_{\max}$ needs to be more than 18 months for $A1$ and $A3$ adjusted idiosyncratic real exchange rates to reject the nonstationary null, and yield $\bar{p}$ no less than 8.9 months. In contrast, when α is 0.05 and $p_{\max}$ is no greater than 18 months, $A2$ adjusted idiosyncratic real exchange rates show no evidence of convergence at all levels; when α is 0.05 and $p_{\max}$ is 24 months, $A2$ adjusted idiosyncratic real exchange rates show weak evidence of convergence at 10% level. We focus on $\alpha = 0.05$ because this is the "typical" critical value

[10] We actually performed $\alpha = \{0.025, 0.05, 0.1, 0.15, 0.2, 0.25, 0.5, 1\}$ but choose to report $\alpha = \{0.05, 0.25, 1\}$ because the estimated result do not vary too much when $\alpha \leq 0.1$ and $\alpha \geq 0.15$ and we use $\alpha = 1$ as a modeling benchmark.

for serial correlation in literature. If we follow this "typical" significant level, we are going to reject the nonstationary null for most of the times.

However, if we are willing to let α to be greater than 0.10, or more "loosely" control the serial correlation of each countries' time series, the evidence of stationarity would be stronger. For α equals to 0.25, $A1$ and $A3$ tend to reject the null after 9 months of maximum potential lag length at 6% level. When we turn to $A2$, we can observe that the evidence of stationary is a little weaker than $A1$ or $A3$. In $A2$ case, we shall reject the unit root null when $\{\alpha \geq 0.25, p_{\max} \geq 12\}$ at 7% level. In other words, the choice of approximation of common feature would modestly modify our testing result.

Horizontally probe table 2.1, we can find that for $\bar{p}$ is less than 6 months, or two quarters, we cannot reject the null for any case. This finding partially confirms Pesaran (2007) testing results. Another interesting finding is that as α increasing, for all alternatives of the common feature, it generally needs less lagged periods to conclude stationarity. This is to say, if current-past dependence of real exchange rates is more "loosely" controlled, we shall need less lagged period to support for real exchange rates convergence.

Finally, when we turn to the benchmark case when $\alpha = 1$, evidence of convergence is the strongest among all priori specification of α. This may be due to the large number of lagged terms we added into the regression. As we forcing all countries to select $p_{\max}$ numbers of months as lags, the loss of freedom is huge. Nevertheless, even the evidence from the benchmark is the strongest, it still require no less than 6 months lags to conclude stationarity.

Table 2.2 reports the estimated half-life and corresponding 95% confidence interval of idiosyncratic real exchange rates convergence after isolating for common factors for period 1973M03 - 1998M12. Notice that we only focus on the half-life with bounded upper 95% confidence interval. For those half-life which have infinite upper bound, Table 2.1 shows that they are statistically insignificant and fail to reject the unit root null. Vertically examine the table, when the common factor is considered as $A1$ (the arithmetic average) or $A3$ (the principal component method estimation), the estimated half-life varies from 34 to 45 months, or 2.83 to 3.75 years, which is around the lower bound of "Rogoff's consensus". If the common factor is considered as $A2$ (the trading weighted average), the estimated half-life varies from 60 to 63 months, or about 5 years, which is the upper bound of "Rogoff's consensus". Our general conclusion is, the idiosyncratic real exchange rates have lower half-life if they are "de-featured" by $A1$ or $A3$ than by $A2$. This result should not be surprising, since as we can see from Figure 1, the estimation of common feature using $A1$ or $A3$ is always more volatile that that of $A2$. Therefore, The "de-featured" idiosyncratic real exchange rates of $A2$ would be more volatile than that of $A1$ or $A3$. Meanwhile, obtaining a lower half-life from $A1$ or $A3$ does not mean that either of them are superior to $A2$, because a lower half-life is *not* the criteria to judge superiority: should the economic reasoning be the criteria instead of empirical results.

Individually, the half-life estimations are decreasing along with maximum lag length $p_{\max}$ increasing. In other words, when we choose more lags, the estimated half-life falls. This result suggests a long-memory of real exchange rates. This also suggests a trade-off between control of serial correlation and estimation of half-life. As we discussed above, choosing more lags could distort the size of our tests; in another hand, it seems that the estimated half-life could be reduced by including more lags.

When we examine horizontally across each significance level, for the same maximum lag length specification $p_{\max}$, the half-lives also decrease when significance level α increase. For a general conclusion, we can reject the unit root null in presence of cross-section dependence only with more lagged terms or "looser" control of size for serial correlation. And even if we reject the nonstationary null, at least for this sampling period, the mean reverting half-life of idiosyncratic real exchange rates is still too long to accommodate the the nominal price rigidity period.

2.5.2 1988M12 - 1998M12

Table 2.3 is the unit root tests for the period 1989M01 - 1998M12 following the reasoning provided by Smith et al. (2004). if we restrict ourselves on critical value of $\alpha = 0.05$, table 2.3 reports extremely weak evidence of real exchange rates convergence. For p_{max} shorter than 18 months, the nonstationary null cannot be rejected in 10% level. In other words, if we adopt typical criteria to control for serial correlation, after adjusting any of the three alternatives of common features, the idiosyncratic real exchange rates are nearly random walk. Furthermore, this conclusion is robust of almost all lag length identifications except for p_{max} equals to 24 months.

Mark and Sul (2008) also focused on post 1988 periods and found similar results as ours. They finally concluded that the LOP does not hold at all when assuming cross-country dependence. However, it is in doubt of the robustness of their empirical results as well as ours. Because, again, the maximum lag length $p_{\max}$ combining with the significance level to control the size for serial correlation α are set arbitrarily chosen. Further examine Table 2.3, when $\alpha = 0.25$, we can reject the unit root null for $p_{\max} \geq 12$ and yield $\bar{p} \geq 7.9$ for all three alternatives of common feature at 5% level. Specifically, it seems that $A2$ is more likely to reject the unit root null than the other two competitors. $A2$ adjusted idiosyncratic real exchange rates tend to be stationary at 10% level only after the p_{max} is greater or equal to 6 months. Setting $\{\alpha \geq 0.25, p_{\max} \geq 6\}$, we got $\bar{p}$ to be 4.75 months. This is to say, the test results are not robust to priori parameter settings of α and $p_{\max}$.

Further evidence are provided by the benchmark where we force $\alpha = 1$. We reject the unit root null for almost all the maximum lag length selection except p_{max} is 3 months. Our finding indicates that individual serial correlation is important of testing unit roots in dynamic panel data structure. When we have appropriately controlled for serial correlation, the power of unit roots test shall be enhanced.

Table 2.4 is the estimated half-life for 1989M01 - 1998M12. The estimation is considerably shorter than period 1973M03 - 1998M12 reported in Table 2.2. This implies potential structural breaks before 1988.[11] For common factor $A2$, the half-life varies from 18 to 24 months, or 1.5 to 2 years. Such low half-life could coordinate a price stickiness period. An important case is when we choose $\alpha = 0.25$ and $p_{\max} = 24$, which is acceptable in practice, for common factor alternative $A2$, which yields $\bar{p} = 15.3$ from Table 2.3, the 95% confidence interval is $[12, 34]$ months. The lower bound touches the price rigidity period which is reported by Chamberlain and Rothschild (1983) as an at-the-dock price stickiness to be 12 months for imports, and 14 months for exports.

Considering all possible parameter specifications, the evidence of long-run convergence is mixed. If we restrictively control for individual serial correlation, the evidence is very weak and we might consider longer lags to adjust potential size distortion of unit root testing. But we also need to find good economic reasons to include more lags. For example, it is hard to understand why Italy needs 10 months lags but Ireland requires 36 lags for both of them to control serial correlation of real exchange rates under the size of 5%.[12] However, the more "loosely" controlling for serial correlation, the stronger evidence of convergence of idiosyncratic real exchange rates can be obtained given a fixed maximum lag length. Overall, we can find that the idiosyncratic real exchange rates indeed converge when we have properly controlled exogenous parameters which are set in priori to be some specific values (and do not change) in literature. Those exogenous parameter values are set generally with little argument of economic reasonings. We have shown that the convergence of idiosyncratic real exchange rates is not robust to those parameter settings.

2.6 Concluding remarks

In this essay we provided mixed evidence of convergence of idiosyncratic real exchange rates. We call it "the numeraire problem" of real exchange rates when there exist cross-sectional

[11] See Smith et al. (2004) for further discussion of potential structural breaks during the period and the reasoning to choose such sub-sample period.

[12] This example is obtained by extending our experiment with $\alpha = 0.05$ and $p_{max} = 48$. The result is skipped and available upon request.

dependence caused by adopting a numeraire. Using a single common factor model, we isolated countries' idiosyncratic effects of real exchange rates from the contamination of cross-country comovement effect (the common feature). We used three alternative estimations of the common feature where the second one is the real effective exchange rates (REER) of the numeraire country. To our best knowledge, this is a newly proposed common feature approximation in real exchange rates literature. The merit of using REER as the common feature is not that it is first used in literature, but the trading weighted it adopted to probe the interactive effects. Obviously it is superior to the arithmetical average of real exchange rates.

The evidence of convergence of idiosyncratic real exchange rates is mixed. Generally speaking, if we "loosely" control for a country's individual serial correlation, the unit roots hypothesis could be rejected with 9 months lags or more. We also studied a sub-sample period for 1988M12 to 1998M12. The half-life of real exchange rates mean reverting period was considerably shorter than what is previously found in literature. For example, if we choose to control of serial correlation of size 25% and use maximum lag length of 18 months, our point estimation of half-life is 18 months, with 95% confidence interval of $[12, 38]$ months. This result is significantly shorter than "Rogoff's consensus" half-life which is 3- to 5-year. This finding also implies that there might be structural breaks before 1988. As a conclusion, the evidence of convergence of idiosyncratic real exchange rates is weaker than previous literature documented. But interestingly, when there is evidence of convergence, the half-life is much shorter. And also, the evidence of convergence is not robust to priori parameter settings.

We finally note that our factor model can be also viewed as a simple state-space model without input variables. The further study could be done in the direction of including input variables which accounts for central banks foreign market intervention. Considering the nature of the state-space model, we predict that the dramatic change of the US dollars in the middle 1980's could be absorbed by the input variables with appropriate parameter identification. Thus the evidence of convergence could be stronger than what is found in this essay.

Table 2.1: Panel Unit Root Test 1973M03-1998M12.

		A1			A2			A3		
α [a]	p_{max} [b]	$\bar{p}$ [c]	δ	$Pr \geq \|t\|$	$\bar{p}$	δ	$Pr \geq \|t\|$	$\bar{p}$	δ	$Pr \geq \|t\|$
0.05	3	2.85	-0.01382	0.20	2.85	-0.00925	0.33	2.85	-0.01408	0.20
	6	3.70	-0.01364	0.16	3.65	-0.00952	0.27	3.70	-0.01385	0.15
	9	3.65	-0.01364	0.16	4.05	-0.00952	0.24	3.70	-0.01385	0.15
	12	4.45	-0.01364	0.12	5.10	-0.01105	0.15	5.00	-0.01385	0.09
	18	8.90	-0.01755	0.02	6.30	-0.00925	0.15	8.85	-0.01813	0.01
	24	11.15	-0.01755	0.01	9.45	-0.00925	0.08	10.55	-0.01813	0.01
0.25	3	2.90	-0.01382	0.20	2.95	-0.00925	0.33	2.90	-0.01408	0.19
	6	4.65	-0.01364	0.11	4.70	-0.00952	0.21	4.50	-0.01385	0.11
	9	6.50	-0.01364	0.06	6.30	-0.00952	0.14	6.10	-0.01385	0.06
	12	10.60	-0.01761	0.01	8.05	-0.01136	0.07	8.85	-0.01808	0.01
	18	15.85	-0.01810	< 0.01	15.25	-0.01085	0.02	15.55	-0.01879	< 0.01
	24	19.60	-0.01810	< 0.01	21.15	-0.01085	0.01	19.40	-0.01879	< 0.01
1	3	3	-0.01382	0.19	3	-0.00925	0.32	3	-0.01408	0.18
	6	6	-0.01332	0.09	6	-0.00909	0.18	6	-0.01369	0.08
	9	9	-0.01499	0.03	9	-0.01017	0.08	9	-0.01543	0.02
	12	12	-0.01761	< 0.01	12	-0.01136	0.03	12	-0.01808	< 0.01
	18	18	-0.01839	< 0.01	18	-0.01078	0.01	18	-0.01920	< 0.01
	24	24	-0.01903	< 0.01	24	-0.01145	< 0.01	24	-0.01988	< 0.01

Note: A1 uses the arithmetic average of cross countries real exchange rates as the estimation of common feature; A2 uses the trading weighted average of cross countries real exchange rates (REER of the numeraire country) as the estimation of common feature; A3 uses principal component method to estimate the common feature.

[a] *α is the size control for serial correlation set in priori.*

[b] *p_{max} is the maximum lag length set in priori.*

[c] *$\bar{p} = \frac{1}{N}\sum_{i=1}^{N} p_i$ where p_i is country i's appropriate lag length order. p_i is determined by General-to-Specific identification.*

Table 2.2: Half-life Estimation 1973M03-1998M12.

		$A1$		$A2$		$A3$	
α [a]	p_{max} [b]	Half-life	$95\%CI$	Half-life	$95\%CI$	Half-life	$95\%CI$
0.05	3	49	[20 , ∞]	74	[25 , ∞]	48	[20 , ∞]
	6	50	[22 , ∞]	72	[27 , ∞]	49	[21 , ∞]
	9	50	[21 , ∞]	72	[28 , ∞]	49	[21 , ∞]
	12	50	[23 , ∞]	62	[27 , ∞]	49	[23 , ∞]
	18	39	[22 , 166]	74	[32 , ∞]	37	[21 , 148]
	24	39	[23 , 123]	74	[36 , ∞]	37	[22 , 119]
0.25	3	49	[20 , ∞]	74	[26 , ∞	48	[20 , ∞]
	6	50	[23 , ∞]	72	[29 , ∞]	49	[23 , ∞]
	9	50	[25 , ∞]	72	[32 , ∞]	49	[25 , ∞]
	12	39	[23 , 120]	60	[30 , ∞]	37	[21 , 134]
	18	37	[23 , 90]	63	[35 , 297]	36	[23 , 84]
	24	37	[24 , 79]	63	[37 , 191]	36	[24 , 74]
1	3	49	[20 , ∞]	74	[26 , ∞]	48	[20 , ∞]
	6	51	[24 , ∞]	75	[31 , ∞]	50	[24 , ∞]
	9	45	[25 , 252]	67	[33 , ∞]	44	[24 , 216]
	12	39	[23 , 106]	60	[33 , 325]	37	[23 , 99]
	18	37	[24 , 81]	63	[36 , 246]	35	[23 , 74]
	24	36	[24 , 70]	60	[36 , 166]	34	[23 , 65]

Note: p_{max}, Half-life and 95% Confidence Interval are all reported in months. A1 uses the arithmetic average of cross countries real exchange rates as the estimation of common feature; A2 uses the trading weighted average of cross countries real exchange rates (REER of the numeraire country) as the estimation of common feature; A3 uses principal component method to estimate the common feature.

[a] *α is the size control for serial correlation set in priori..*

[b] *p_{max} is the maximum lag length set in priori.*

Table 2.3: Panel Unit Root Test 1988M01-1998M12.

		A1			*A2*			*A3*		
α [a]	p_{max} [b]	$\bar{p}$ [c]	δ	$Pr \geq \|t\|$	$\bar{p}$	δ	$Pr \geq \|t\|$	$\bar{p}$	δ	$Pr \geq \|t\|$
0.05	3	2.65	-0.02015	0.29	2.7	-0.02623	0.23	2.7	-0.02302	0.24
	6	3.15	-0.02015	0.25	3.85	-0.02623	0.15	3.15	-0.02302	0.20
	9	3.15	-0.02015	0.25	3.75	-0.02623	0.16	3.15	-0.02302	0.20
	12	3.1	-0.02015	0.25	3.65	-0.02623	0.16	2.9	-0.02302	0.22
	18	4.6	-0.02015	0.16	3.5	-0.02623	0.17	4.55	-0.02302	0.12
	24	8.85	-0.02015	0.05	6.25	-0.02762	0.05	8.15	-0.02302	0.04
0.25	3	2.75	-0.02015	0.28	2.8	-0.02623	0.22	2.75	-0.02302	0.23
	6	4.25	-0.02222	0.15	4.75	-0.03022	0.08	4.1	-0.02577	0.11
	9	5.35	-0.02219	0.10	5.5	-0.03022	0.06	5.45	-0.02472	0.07
	12	8.15	-0.02903	0.01	7.9	-0.03638	0.01	8.55	-0.03149	0.01
	18	11.15	-0.02903	< 0.01	12.2	-0.03638	< 0.01	11.95	-0.03149	< 0.01
	24	16.95	-0.02686	< 0.01	15.3	-0.03638	< 0.01	18.45	-0.02904	< 0.01
1	3	3	-0.02015	0.26	3	-0.02623	0.21	3	-0.02302	0.21
	6	6	-0.02222	0.09	6	-0.03022	0.05	6	-0.02577	0.05
	9	9	-0.02638	0.01	9	-0.03743	0.00	9	-0.02840	0.01
	12	12	-0.02903	< 0.01	12	-0.03516	< 0.01	12	-0.03149	< 0.01
	18	18	-0.03015	< 0.01	18	-0.03196	< 0.01	18	-0.03167	< 0.01
	24	24	-0.03430	< 0.01	24	-0.03200	< 0.01	24	-0.03663	< 0.01

Note: A1 uses the arithmetic average of cross countries real exchange rates as the estimation of common feature; A2 uses the trading weighted average of cross countries real exchange rates (REER of the numeraire country) as the estimation of common feature; A3 uses principal component method to estimate the common feature;

[a] *α is the size control for serial correlation set in priori..*

[b] *p_{max} is the maximum lag length set in priori.*

[c] *$\bar{p} = \frac{1}{N}\sum_{i=1}^{N} p_i$ where p_i is country i's appropriate lag length order. p_i is determined by General-to-Specific identification.*

Table 2.4: Half-life Estimation 1988M01-1998M12.

		$A1$		$A2$		$A3$	
α	p_{max}	Half-life	$95\% CI$	Half-life	$95\% CI$	Half-life	$95\% CI$
0.05	3	34	$[13 , \infty]$	26	$[10 , \infty]$	29	$[12 , \infty]$
	6	34	$[13 , \infty]$	26	$[11 , \infty]$	29	$[12 , \infty]$
	9	34	$[13 , \infty]$	26	$[11 , \infty]$	29	$[12 , \infty]$
	12	34	$[13 , \infty]$	26	$[11 , \infty]$	29	$[12 , \infty]$
	18	34	$[15 , \infty]$	26	$[11 , \infty]$	29	$[14 , \infty]$
	24	34	[18 , 222]	24	[13 , 157]	29	[16 , 149]
0.25	3	34	$[13 , \infty]$	26	$[10 , \infty]$	29	$[12 , \infty]$
	6	30	$[14 , \infty]$	22	$[11 , \infty]$	26	$[12 , \infty]$
	9	30	$[15 , \infty]$	22	[11 , 174]	27	$[14 , \infty]$
	12	23	[14 , 70]	18	[11 , 52]	21	[13 , 56]
	18	23	[14 , 54]	18	[12 , 38]	21	[14 , 45]
	24	25	[16 , 50]	18	[12 , 34]	23	[16 , 42]
1	3	34	$[13 , \infty]$	26	$[11 , \infty]$	29	$[12 , \infty]$
	6	30	$[15 , \infty]$	22	$[12 , \infty]$	26	$[14 , \infty]$
	9	25	[15 , 79]	18	[11 , 42]	24	[14 , 67]
	12	23	[15 , 52]	19	[12 , 42]	21	[14 , 45]
	18	22	[15 , 39]	21	[13 , 44]	21	[15 , 36]
	24	19	[14 , 30]	21	[14 , 40]	18	[13 , 28]

Note: p_{max}, Half-life and 95% Confidence Interval are all reported in months. A1 uses the arithmetic average of cross countries real exchange rates as the estimation of common feature; A2 uses the trading weighted average of cross countries real exchange rates (REER of the numeraire country) as the estimation of common feature; A3 uses principal component method to estimate the common feature.

[a] *α is the size control for serial correlation set in priori..*

[b] *p_{max} is the maximum lag length set in priori.*

Chapter 3

Constructing the theoretical monetary aggregation for Chinese Renminbi

During the past three decades, the undergoing market-oriented economic reforms have significantly changed China's financial market structure. For instance, RMB foreign exchange rate has been gradually moving toward a more flexible regime, RMB interest rate has been liberalizing and the capital markets such as stock markets and bond markets have experienced rapid growth. Such quick transformations of financial system posed great challenges for conducting monetary policy in China. To be compatible with fast reformations, the PBC is consistently changing its policies to control China's economy along with fiscal policy conducted by China's treasury department. At the third quarter of 1994, People's Bank of China started to announce the balance of money supply to public. In January 1998, the PBC officially announced that it would use "market based" policy tools to carry out the monetary policy.[1]

When conducting monetary economics research, most literatures use the simple-sum monetary aggregation as the indicator of money supply. However, Barnett (1980) proved that the simple-sum money aggregate is incompatible with microeconomics foundations built

[1] The "market based" policy tools consist of open market operations, discount rates, reserves ratios, liquidity and capital requirements, and etc.

on the hypothesis that rational representative agent optimizes her utility. Based on the assumptions of linear homogeneity and weak separability of the utility function, Barnett (1980) developed rigorous formula of the user cost of a financial asset. Barnett's view of inconsistency of simple-sum monetary aggregation with the microeconomics theory is further supported by fruitful empirical work, which include but not limiting to Barnett (1984, 1997), Belongia (1996), and Anderson et al (1997a, 1997b). Furthermore, the theoretical monetary aggregate, which is called Divisia index, has been applied to the United Kingdom (Fisher et al. 1993), Japan (Ishida and Nakamura, 1993), Canada (Longworth and Atta-Mensah, 1994), and the United States (Anderson et al. 1997a, 1997b).

3.1 Background: China's money supply and monetary policy

In this essay, we construct the Divisia index of China money supply using the data from January 1996 to December 2009, totally 168 month. Renminbi (RMB) is the official fiat money of People's Republic of China, and People's Bank of China (PBC) is China's central bank. During the past three decades, the undergoing market-oriented economic reforms have significantly changed China's financial market structural. Specifically, RMB foreign exchange rate is gradually moving toward a more flexible regime, RMB interest rate is being liberalizing and the rapid growth of capital markets such as stock markets and bond markets. Such quick transformations of financial system posed great challenges for conducting monetary policy in China. To be compatible with fast reformations, the PBC is consistently changing its policies to control China's economy along with fiscal policy conducted by China's treasury department. Since the third quarter of 1994, People's Bank of China started to announce the money supply quantity to public. In January 1998, the PBC officially announced that it would use "market based" policy tools to carry out the monetary policy.[2]

Whether the money demand is "stable" is an important question, since the central bank conducts the monetary policy based on, fundamentally, the money demand. In case of China, considering the rapid development of the economy and fast reformation of the financial system, such question is also interesting and should be considered in priority before studying China's monetary policy. Most literature studying China's money demand used the simple-sum method for the monetary aggregation. However, Barnett (1980) asserted that the simple-sum money aggregates is incompatible with microeconomics foundations build on the hypothesis that rational representative agent optimizes her utility. Based the assumptions of linear homogeneity and weak separability of the utility function, Barnett (1980) developed rigorous formula of user cost of financial assets which provide pure financial liquidity service. Barnett's view of inconsistency of simple-sum monetary aggregation with the microeconomics theory is further supported by fruitful empirical work, which include but not limiting to Barnett (1984, 1997), Belongia (1996), and Anderson et al (1997a, 1997b). Furthermore, the Divisia monetary aggregates index has been applied to the United Kingdom (Fisher et al. 1993), Japan (Ishida and Nakamura, 1994), Canada (Longworth and Atta-Mensah, 1994), and the United States (Anderson et al. 1997a, 1997b).

This essay is devoted into constructing of the accurate monetary aggregation of China's RMB.[3] Yu and Tsui (2000) was the first attempt to construct China's monetary aggregation using Divisia index, for which they referred as Monetary Service Index (MSI) following Anderson et al., (1997a, 1997b). Their data set consisted of 168 monthly observations from January 1984 to December 1997, which is extracted from Monthly Statistics of China published by the State Commission of Statistics of China. However, the rapid economy growth and quick financial innovations implies potential structural changes of China's monetary demand, thus the up-to-date version of Divisia index of RMB is needed for further study

[2]The "market based" policy tools consist of open market operations, discount rates, reserves ratios, liquidity and capital requirements, and etc.

[3]We call such accurate monetary aggregation the "Divisia index", which will be fully illustrated in the following section.

purpose.[4]

3.2 The model of theoretical monetary aggregates

We follow Barnett (1978, 1980) to assume that the representative agent maximizes her inter-temporal utility function under a budget constraint which contains both consumption commodities and monetary assets over a finite horizon. By assuming that the utility function is linear homogeneous and weakly separable in the group of current period monetary assets against the group of commodities, the representative agent is able to formulate a two stage decision. In the first stage she is deciding to split her income into two blocks, one block of purchasing consumption commodities and the other one of investing into monetary assets. Then, in the second stage, she is able to directly maximize her sub-utility under the constraint of monetary assets. To be specific, the representative agent's inter-temporal optimization problem of the second stage can be formulated as the following:

$$\max U = U(m_{1t}, ..., m_{Nt})$$

subject to the budget constraint

$$\sum_{i=1}^{N} m_{it}\pi_{it} = y_t.$$

where m_{it} , $i = 1, ..., N$, are the balance of different monetary assets she holds for period t. Notice that the agents disposable income of monetary asset, y_t, is pre-determined by her first stage decision so that y_t is a proportion of her total income at period t. π_{it}'s are the user cost

[4] By financial innovations we briefly concerning on three aspects. The first thing is the interest rates liberalization of China. Since 1996, the bond market rate, the interbank market rate and partially the commercial banks' saving account rates are liberalized. The second thing is the reform of RMB's foreign exchange rate regime. On July 21st, 2005, the PBC officially announced that it will adopt a managed floating regime of RMB aiming on a basket of currencies. During the period of 2005-2008, the RMB real effective exchange rate appreciated around 14%. The third one is the improvement of capital markets of China. Several kinds of mutual funds have been developed in China sine 1998. For example, Wang and Yu (2010) used a disaggregated survey data which showed that during 2007 Chinese urban residence distributed 34% of their total household saving into stock (by 11.28%), mutual funds (5.52%), and bonds (2.83%).

of holding monetary asset i at time t. The user cost can be interpreted as the opportunity cost of holding a monetary asset. Our agent forgoes a higher interest if she keeps holding monetary asset i instead of convert it into a higher interest bearing monetary asset. Further notice that by considering the user costs of holding the monetary assets, we are actually regarding the monetary assets as durable goods, which mean that the monetary assets do not fully depreciate within the decision period. This is reasonable since monetary assets, for example the demand deposits, provide not only the store of value but also the service of liquidity. In other words, if some monetary asset, say, the currency in circulation, provides only service of liquidity, it is non-durable and fully depreciate by the end of the current period. If some monetary asset, say, long term deposit with large amount, provides only store of value (by assuming that our agent will not withdraw it before maturity date), it is durable and fully depreciate only by the end of its own maturity date, which is multi-period after the current decision period. Thus, the smaller the user cost is, the closer the monetary asset is as a substitute of a monetary asset which is functioning purely as a store of value.

The user cost of a monetary asset is the key of constructing of the theoretical monetary aggregation. Let r_{it} denote the nominal rate of return of monetary asset i during period t, and let R_t be the benchmark asset's nominal rate of return. The benchmark asset should be risk-free and functioning purely as a store of value. The usage of the benchmark asset is only for inter-temporally transferring of wealth and it provides no service of liquidity. Then Barnett (1980) rigorously developed the formula of user cost to be

$$\pi_{it} = \frac{R_t - r_{it}}{1 + R_t} \tag{3.1}$$

which is the discounted interest forgone by holding the monetary asset i instead of holding the benchmark asset by the end of period t.

Following Anderson et al. (1997a, 1997b), we prudently distinguish the nominal balance of monetary assets and the real balance of monetary assets, as well as the nominal and real user costs. Let $m_{1,t}^{nom}$ denote the *optimal* nominal balance of monetary asset i, and let

$$m_t^{nom} = (m_{1,t}^{nom}, ..., m_{N,t}^{nom})$$

denote the vector of all *optimal* nominal monetary assets. Similarly, let $m_{1,t}^{real}$ denote the *optimal* real balance of monetary asset i, and let

$$m_t^{real} = (m_{1,t}^{real}, ..., m_{N,t}^{real})$$

denote the vector of all optimal real monetary assets. The relation of nominal and real balance of monetary assets is given by

$$m_{i,t}^{real} = m_{it}^{nom}/p^*,$$

where p^* is a true cost-of-living index.[5]

The real user cost of monetary asset i at time t is given by equation (3.1) and the corresponding nominal user cost is given by

$$\pi_{it}^{nom} = p^* \frac{R_t - r_{it}}{1 + R_t}. \tag{3.2}$$

Therefore the nominal user cost and real user cost is related by the identity

$$\pi_{it}^{real} = \pi_{it}^{nom}/p^*.$$

The real balance of a monetary asset multiply by its nominal user cost is equal to the total expenditure on that asset. Thus the representative agent's total expenditure on monetary

[5]There are different cost-of-living indexes for different research purposes. If we focus on the consumer decisions, then we could use the Consumer Price Index (CPI). If we focus on the firms or banks, the Producer Price Index (PPI), Retail Price Index (RPI) or Wholesale Price Index (WPI) could be appropriate. We can also use the GDP deflator if we do not specify the supply side or demand side.

asset can be obtained by

$$\begin{aligned} y_t &= \sum_{i=1}^{N} \pi_{it}^{nom} m_{it}^{real} \\ &= \sum_{i=1}^{N} \pi_{it}^{real} m_{it}^{nom} \end{aligned} \tag{3.3}$$

3.3 Divisia index of theoretical monetary aggregation

Barnett (1980, 1990) showed that under certain conditions, the theoretical monetary aggregation can be represented by a class of superlative statistical index numbers. Superlative statistical index number, which is a non-parameter methodology, was first discussed in Diewert (1976). Diewert defines that a statistical index number to be "superlative" if it is exact for some aggregation function which can provide a second-order approximation to an arbitrary linear homogeneous aggregation function in discrete time. Among many superlative index numbers, Törnqvist (1936) and Theil (1967) advocated the following quantity index number, called the Törnqvist-Theil index

$$M_t = M_{t-1} \prod_{i=1}^{N} \left(\frac{m_{i,t}}{m_{i,t-1}} \right)^{(s_{it}+s_{it-1})/2}$$

where $s_{it} = \pi_{it} m_{it} / \sum_{j=1}^{N} \pi_{kt} m_{kt}$ is the expenditure share of monetary asset i. Take logarithm of both side we obtain

$$\Delta \log M_t = \sum_{i=1}^{N} s_{it}^{*} \Delta \log m_{it}$$

where $s_{it}^{*} = (s_{it} + s_{it-1})/2$ and $\Delta X_t = X_t - X_{t-1}$ denotes the first difference. Törnqvist-Theil index is known to retain its second-order tracking properties when some common aggregation theoretic assumptions are violated. Because of its connection with Divisia's (1925) continuous time index number, Barnett (1980) referred the Törnqvist-Theil index as the Törnqvist-Theil Divisia Index. We will call it *Divisia index* for convenience. Anderson et al (1997a, 1997b, 1997c) referred Törnqvist-Theil index as Monetary Service Index (MSI).

A price index (Divisia user cost index) is said to be *dual* to its quantity index (Divisia quantity index) if their product equals the agent's total expenditure amount on monetary assets. Such property is called factor reversal in Fisher (1922). To be specific, Divisia quantity index (M_t) and its dual user cost index (Π_t) can be expressed as

$$\Pi_t \cdot M_t = y_t = \sum_{i=1}^{N} \pi_{it} m_{it}.$$

The dual user cost can be interpreted as an comprehensive price of money. We can intuitively consider this dual user cost as a (weighted) average (aggregate) of interest rates (r_{it}) of different monetary assets. Among years, there are other attempts tried to aggregate interest rates. For example, Wong et al. (2005) proposed to construct the "composite rate" using weighted average for the Hong Kong dollar interest rate. Their composite rate of Hong Kong dollar is given by

$$\Pi_t^{Composit} = \frac{\sum_{i=1}^{N} r_{it} m_{it}}{\sum_{i=1}^{N} m_{it}},$$

which is the average of interest rates weighted by the balance of corresponding components.[6] Nevertheless, we need our dual price index to be Diewert-superlative. Fisher (1922) gave a weak factor reversal criterion of the dual user cost by the formula

$$\Pi_t = \Pi_{t-1} \left(\frac{y_t / y_{t-1}}{M_t / M_{t-1}} \right).$$

Notice that Divisia quantity index number and its dual user cost index are constructed as chained Diewert-superlative indexes, so they retain the same statistical properties as other chained superlative quantity and price indexes, such as gross domestic product (GDP) and consumer price index (CPI).

[6] Due to the government confidential problem, we are unable to obtain the composite rate of Hong Kong dollar for research purpose. The composite rate of Hong Kong dollar is only available for the internal use of Hong Kong Monetary Authority.

Since we distinguish the nominal and real balance of monetary assets as well as their user costs, we need also distinguish nominal and real Divisia quantity index and their dual user costs. Let M_t^{nom} and M_t^{real} denote the nominal and real Divisia quantity indexes, respectively. Let Π_t^{nom} and Π_t^{real} denote the nominal and real dual user cost indexes, respectively. We define the nominal Divisia quantity index by formulating its growth rate

$$\Delta \log M_t^{nom} = \sum_{i=1}^{N} \frac{(s_{it}^{nom} + s_{it-1}^{nom})}{2} \Delta \log m_{it}^{nom}, \tag{3.4}$$

where $s_{it}^{nom} = \pi_{it}^{real} m_{it}^{nom} / \sum_{j=1}^{N} \pi_{kt}^{real} m_{kt}^{nom}$. The user cost index dual to nominal Divisia quantity index is given by

$$\Pi_t^{real} = \frac{y_t}{M_t^{nom}} = \frac{\sum_{i=1}^{N} \pi_{it}^{real} m_{it}^{nom}}{M_t^{nom}}. \tag{3.5}$$

Similarly, we define the real Divisia quantity index and its dual nominal user cost index by changing the superscripts,

$$\Delta \log M_t^{real} = \sum_{i=1}^{N} \frac{(s_{it}^{real} + s_{it-1}^{real})}{2} \Delta \log m_t^{real}, \tag{3.6}$$

where $s_{it}^{real} = \pi_{it}^{nom} m_{it}^{real} / \sum_{j=1}^{N} \pi_{kt}^{nom} m_{kt}^{real}$. The user cost index dual to nominal Divisia quantity index is given by

$$\Pi_t^{nom} = \frac{y_t}{M_t^{real}} = \frac{\sum_{i=1}^{N} \pi_{it}^{mon} m_{it}^{real}}{M_t^{real}}. \tag{3.7}$$

3.4 Chinese Renminbi and corresponding interest rates

3.4.1 RMB official aggregations and the data defects

The official definition of RMB aggregate supply is using the simple-sum method and is divided into four categories:

Definition 1 *RMB Official (theoretical) aggregation*

- M0: currency in circulation;
- M1: M0 + demand deposit of enterprise + deposits of public institutions[7] + deposit of government department and organizations + rural deposit + individual credit card type deposit[8];
- M2: M1 + total deposit of resident sector + time deposit of enterprise + foreign currency deposit + trust deposit;
- M3: M2 + financial bonds + business papers + negotiable CDs.

Monthly data of M0, M1 and M2 on core indicators and credit aggregates are released in the daily newspaper (China) “Financial Times” 2-4 weeks after the end of the reference month. Monthly data on analytical accounts of monetary authorities, banking institutions, and Depository Corporations Survey are released on PBC’s official web site 4 weeks after the end of the reference month.

China joined the IMF’s General Data Dissemination System (GDDS) on April 15th, 2002. Since then, the PBC started to establish its metadata of GDDS on its official web site. According to PBC’s GDDS metadata Table B (Data Categories and Indicators, Financial Sector), section I (Data Characteristics),

> “...depository corporations survey includes monetary authorities (People’s Bank of China, PBC) and other depository corporations. Beginning in 2002, depository corporations survey includes foreign banks in China and foreign currency accounts of local banks. Since 2002, the People’s Bank of China has revised the system of monetary and financial statistics in line with the IMF Manual on Mon-

[7] As an explicitly announced Socialism country, China has a large number of public institutions, such as hospitals, electricity companies, railway transportation and urban metros.

[8] It is a little wired to deposit savings in credit cards. However, considering the characteristics of Chinese people habits, such as that people prefer to save-then-consume other than borrow-and-consume, it is understandable that Chinese people deposit currency into their credit card account so that they do not have to borrow from the banks.

etary and Financial Statistics. As from 2002, the monetary statistics thereafter would not be fully compatible with historical statistics.[9]"

PBC's GDDS metadata Table B, section I, Coverage, provides a (simplified) applicable definition of the money aggregation of RMB:

Definition 2 *RMB GDDS metadata (Applicable) aggregation*

- M0: the Currency in circulation;
- M1: M0 adding demand deposits (DD) of resident sectors other than central government and banking institutions. ;
- M2: M1 adding time, savings, and other deposits (collectively called "term deposit", TD) of resident sectors other than central government and banking institutions.

People's Bank of China defines the "money" as M1, which is the currency in circulation (CC) added the demand deposits (DD). The "broad money" is defined as M2. "quasi-money" is referred to the term deposit (TD), which is obtained by the simple-sum of time deposits, savings deposits, other deposits and money, from those components constitute the major part of M2 subtract M1. M3 only serves as a conceptual definition for financial innovations, and is not provided by the PBC at all.

Our main resources of the clustered categories of quantity data on deposits are obtained from the People's Bank of China and Financial Yearbook of China. Our data set contains totally 168 months of observation, starting from January 1996 and ending by December 2009. Among piles of statistical report forms, we choose to mainly rely on the following: the "Depository Corporations Survey" (Form S05), the "Sources and Uses of Credit Funds

[9]The revision includes the following 3 aspects: a) to adjust the reporting institutions, i.e. foreign-funded banks being included in the reporting institutions, policy banks being removed from the group of state commercial banks and the separation of urban commercial banks from urban credit cooperatives; b) to expand statistical coverage with foreign exchange business activities of domestic financial institutions being included in monetary statistics; and c) to correct the existing errors and omissions, e.g. the reclassification of rediscount as borrowing from central bank other than the subtraction of claims on other sectors. Resources: Notes of the Sheet of Monetary Survey, PBC web site release, July 5th, 2004.

of Financial Institutions (RMB)" (Form S03), the "Sources and Uses of Credit Funds of Financial Institutions (by sector)" (Form S03a) and the "Money Supply" (Form S07). There are at least three reasons that assure us mainly rely on those four forms and ignore the rest of the forms. First, the "Sources and Uses of Credit Funds of Financial Institutions (RMB)" (Form S03) and the "Depository Corporations Survey" (Form S05) are fully provided by the PBC over our research period. Other forms, which mostly last no more than eight year from the recent release, shall limit our database and potentially bias our measuring. We have to include the "Money Supply" (Form S07) because we need to compare this simple-sum money supply with our newly constructed Divisia quantity index.

A few more clarification which should be highlighted regarding to the definition of the demand deposits (DD), time deposits (TD) and savings deposits (SD), because these three definitions of the PBC is significantly distinguishable form classical concepts. First, demand deposit in China yields explicit interest rates. Second, we observed that the PBC categorized the residents' (individuals') demand deposit into M2, meanwhile the Demand Deposit in M1 only includes the demand deposit made by enterprise. By PBC's definition, Demand Deposit is the "demand deposits of resident sectors other than central government and banking institutions", which is also defined as Money. In the PBC's definition, we find that the position of Demand Deposit in M1 only include demand deposits of enterprise, public institutes total deposit, government department/organization/military total deposit, rural (credit unions) total deposit, and individual credit card type deposit, but excluding the individual residents' demand deposit. However, while yielding much lower interest rate than the term deposits, demand deposit should be included into M1, no matter it is made by individuals or enterprise. We then constructed an "Adjusted M1" for RMB's supply over the period. Figure 3.1 gives the PBC published monetary aggregation and adjusted M1.

The reason we use the "Sources and Uses of Credit Funds of Financial Institutions (by sector)" (Form S03a) is the data statistical discrepancy. We commonly found numerous statistical calculation errors. These statistical errors appear in the forms almost for every

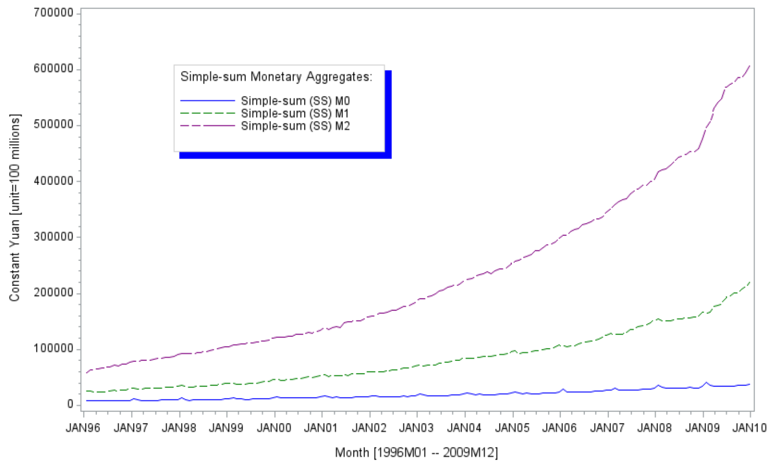

Figure 3.1: The Simple-sum Aggregation of Chinese Renminbi, M0, M1, and M2.

month during our interesting period. Although the PBC admits that there could be some kinds of statistical discrepancy, we think the countless statistical errors shall harm the accuracy of Divisia Index. We will explain this data defects in detail and give two examples of the statistical inaccuracy.

3.4.2 Data defects: example I

By the end of January 2008, the enterprise demand deposit is 88,497.66, the public institutions total deposit is 15,093.42, and government organizations total deposit is 18,421.69, rural deposit is 9,301.24, and no report for individual credit card type deposit. Sum four numbers together will give 131,314.01. Meanwhile the Demand Deposit reported in M1 by that time is 118,197.01, then we summarize that using the category sum-up method would serve a gap of -13,117, for monthly data of January, 2008.

Table 3.1 illustrates this violation of accounting principle. The significant difference of M1 using sum-up method and the numbers reported by the central bank should be highlighted prudently. According to accounting principle, the total numbers should be exactly equivalent. However, the statistical criteria, the statistical errors and billions of reasons may contribute

Table 3.1: Example I: Data discrepancy of M1

Resources of data			
Reported in Form S03a	enterprises demand deposit	88497.66	
	public institutes deposit	15093.42	
	government departments deposit	18421.69	
	rural deposit	9301.24	
	individual credit card type deposit	n/a	
	Total		131314.01
Reported in Form S05	Demand Deposit in M1		118197.01
	Statistical discrepancy		-13117.00

Note: all numbers are in the unit of 100 million Chinese Yuan.
Data resources: People's Bank of China, January 2008 report forms

for the difference. Since we are constructing a Divisia Index of money supply and need to compare it with the simple-sum money supply, we choose to use the results calculated using sum-up method as the M1, and ignore the "official" M1 reported in the form S05. However, we have to use the disaggregated components of sum-up method to construct our Divisia Index. Thus, we ensured the consistency of comparison between the simple sum index and the Divisia index. We also report the comparison of sum-up method with the numbers reported in Form S05 in the next section.

Also keep in mind that the numbers we use for total deposits of public institutions and government departments include both demand deposit and time deposit, and we can assume the Demand Deposit in Form S05 may only includes the organizations' demand deposit, thus the numbers we used to add up for the demand deposits may be upward biased, though by definition we should include all of them. Unfortunately, the demand deposit and time deposit of thus organizations are not published, thus we have no way to confirm our hypothesis. Nevertheless public institutions and government departments may prefer to store their money in demand deposits much more than in time deposits, but we see no reason to exclude the possibility of holding time deposits of such organizations.

3.4.3 Data defects: example II

The components in M2 except M1 is defined as "Quasi Money", which includes Time Deposit, Savings Deposit and Other Deposit. Time Deposit is defined as "the deposits with time deposit characteristic among the enterprise deposits", Savings Deposit is defined as "individual residence's saving deposits", and Other Deposit consists of "foreign deposits" and "trust deposits". It should be highlighted again that Savings Deposit only consists both individual residence's demand deposit and term deposit, and Time Deposit is a terminology regarding to the enterprise only. For example, by the end of January 2008, enterprise time deposits position is 64,020.14, and Time Deposit in Quasi Money reported in the money supply is 65,537.40. Another example serves for the Savings Deposit. By January 2008, the household savings deposits position is 174,304.23 with demand deposit of 67,966.58 and term deposit of 106,337.65. However, when we look up into Form S03a, which is similar to Form S03 but category by sectors, we find that household savings deposits position is 177,412.47 with demand deposit of 67,770.80 and term deposit of 109,641.67. The Savings Deposit reported in Quasi Money is 174,347.88. Slight difference appears again for the exact same category but from different forms. We have found a bunch of this kind of statistical data defects in PBC's official report forms. We consider these data defects would trivially harm our Divisia Index. table 3.2 illustrates this statistical error from different forms.

Table 3.2: Example II: Data discrepancy of quasi-money.

	Form S05	Form S03	Form S03a
Savings Deposit reported in Quasi Money	174,347.88		
household savings deposits position		174,304.23	177,412.47
Demand deposit		67,966.58	67,770.80
time deposit		106,337.65	109,641.67

Note: all numbers are in the unit of 100 million Chinese Yuan.
Data resources: People's Bank of China, January 2008 report forms

We see no reasonable claims to summarize individual sector's demand deposit in Quasi

Money, because by its characteristic, Quasi Money is in the broad money aggregation and have less liquidity than M1, which includes enterprise demand deposits. Generally the transactions of individual sector's demand deposits may be much less comparing to that of enterprise' demand deposits, but demand deposits of individual sectors surely serve as a temporary method of value store. However, when we construct the Divisia price and quantity Index for M2 of RMB, we will use the correspondent demand deposit interest for the individuals' demand deposits and compare the index directly with the simple-sum M2. Considering in the simple-sum assigns each component of M2 equally weight, this example also emphasis the advantage of Divisia Index comparing to the traditional simple-sum method.

3.4.4 Adjusting the interest rates of monetary assets

Basically, the monetary assets included in the People's Bank of China's monetary aggregation can be divided into three categories: currency in circulation (M0), which yield zero interest; money (M1 subtract M0), which yield demand deposit interest; and quasi-money (M2 subtract M1), which yield term deposit interest. Let $r_{0,t}$ denote the interest rate on demand deposit at time t, where $t = 1, ..., T$. Let $r_{n,t}$ denote the interest rate of a term deposit with n month maturity at time t, hence let $r_{3,t}, r_{6,t}, r_{12,t}, r_{24,t}, r_{36,t}$ and $r_{60,t}$ denote 3-month, 6-month, 1-year, 3-year and 5-year term deposit interest rates at time t, respectively. Table 3.3 gives the official deposit interest rates of Chinese Renminbi from 1996 until now. It is shown in the table that the PBC sought to keep the interest rates unchanged over months.

We made two adjustment on the interest rates data. First we convert the annual effective yields on monetary assets into annualized 1-month holding period yields. Let $\widetilde{r}_{n,t}, n = 3, 6, 12, 24, 36, 60$, be the adjusted interest rate at time t. The converting formula is given by Anderson et al. (1997b):

$$\widetilde{r}_{n,t} = \left[\left(1 + \frac{(r_{n,t}/100)}{365}\right)^{30} - 1\right] \times \left(\frac{365}{360}\right) \times 100, n = 3, 6, 12, 24, 36, 60. \tag{3.8}$$

Table 3.3: Chinese Renminbi Official Interest Rates

Date	Demand Deposit (r_t^{dd})	Term deposit (r_t^{td})					
		r_t^{3mon}	r_t^{6mon}	r_t^{1yr}	r_t^{2yr}	r_t^{3yr}	r_t^{5yr}
1996-1-1	3.15	6.66	9.00	10.98	11.70	12.24	13.86
1996-5-1	2.97	4.86	7.20	9.18	9.90	10.80	12.06
1996-8-23	1.98	3.33	5.40	7.47	7.92	8.28	9.00
1997-10-23	1.71	2.88	4.14	5.67	5.94	6.21	6.66
1998-3-25	1.71	2.88	4.14	5.22	5.58	6.21	6.66
1998-7-1	1.44	2.79	3.96	4.77	4.86	4.95	5.22
1998-12-7	1.44	2.79	3.33	3.78	3.96	4.14	4.50
1999-6-10	0.99	1.98	2.16	2.25	2.43	2.70	2.88
2002-2-21	0.72	1.71	1.89	1.98	2.25	2.52	2.79
2004-10-29	0.72	1.71	2.07	2.25	2.70	3.24	3.60
2006-8-19	0.72	1.80	2.25	2.52	3.06	3.69	4.14
2007-3-18	0.72	1.98	2.43	2.79	3.33	3.96	4.41
2007-5-19	0.72	2.07	2.61	3.06	3.69	4.41	4.95
2007-7-21	0.81	2.34	2.88	3.33	3.96	4.68	5.22
2007-8-22	0.81	2.61	3.15	3.60	4.23	4.95	5.49
2007-9-15	0.81	2.88	3.42	3.87	4.50	5.22	5.76
2007-12-21	0.72	3.33	3.78	4.14	4.68	5.40	5.85
2008-10-9	0.72	3.15	3.51	3.87	4.41	5.13	5.58
2008-10-30	0.72	2.88	3.24	3.60	4.14	4.77	5.13
2008-11-27	0.36	1.98	2.25	2.52	3.06	3.60	3.87
2008-12-23	0.36	1.71	1.98	2.25	2.79	3.33	3.60

Resource: People's Bank of China.

For the second adjustment we try to subtract a term premium from the term deposit interest rates. It is obvious from the term structure theory of interest rates that different maturity monetary assets have different term premiums, and hence are not directly comparable. Therefore, the term premiums should be removed from the adjusted interest rates before we use them to construct the user cost of monetary assets. Thus we further adjust the interest rates of monetary assets by subtracting an estimated liquidity premium obtained from the yield curves of "China's treasury bond (fixed return)". The data is obtained from China Bond Information Network.[10] However, the network only provide China's treasury bond yield curves daily data from 2006 until now. Hence we use the following method to estimate the term premiums of China's treasury bond before 2006. This method is modified

[10] Its web site is www.ChinaBond.com.cn, which is in Chinese language.

from Anderson et al. (1997b).[11] Let $r^B_{n,t}$ be the rate of return of treasury bond which mature in n months at time t, and let $r^B_{1,t}$ be one-month secondary-market treasury bond yield obtained from the its yield curve at time t. Then the term premium of any monetary asset is estimated by the term premium of treasury bond, $(r^B_{n,t} - r^B_{1,t})$, since the treasury bond has no default risk. Hence the yield curve adjusted interest rate of an arbitrary term deposit is obtained by

$$r^{YCA}_{n,t} = \widetilde{r}_{n,t} - (r^B_{n,t} - r^B_{1,t}),$$

where $\widetilde{r}_{n,t}$ is the annualized one-month holding-period yield of a term deposit adjusted from (3.8), and $n = 3, 6, 12, 24, 36, 60$. Rearrange the above equation we get

$$r^{YCA}_{n,t} = \left(1 - \frac{r^B_{n,t} - r^B_{1,t}}{\widetilde{r}_{n,t}}\right) \widetilde{r}_{n,t}. \tag{3.9}$$

Due to the lack of data on treasury bond yield curve, we need an alternative approximation of the ratio $\left(r^B_{n,t} - r^B_{1,t}\right)/\widetilde{r}_{n,t}, t = 1, ..., T$. Notice that empirically (i) the interest rates on demand deposit and term deposit are fixed for most periods in our sample, and (ii) the comovement of interest rates is pretty strong, since those interest rates are regulated by the central bank as a part of "financial market stablization". Hence we seek to approximate the ratio $\left(r^B_{n,t} - r^B_{1,t}\right)/\widetilde{r}_{n,t}$ using a function of interest rates on term deposits.

In the term structure theory, the rate of return of a monetary asset with n-month maturity at time t can be approximated by the average of n months expected 1-month yield from period t to period $t + n - 1$, plus a premium term:

$$\widetilde{r}_{n,t} = \frac{1}{n} \sum_{j=0}^{n-1} \widetilde{r}^e_{1,t+j} + prem_{n,t}, \tag{3.10}$$

where $n = 3, 6, 12, 24, 36, 60$ and $t = 1, ..., T - n + 1$. $\widetilde{r}^e_{1,t+j}$ is the expectation of adjusted one-month yield on an artificial term deposit. Since the premium term in our context is

[11] See Anderson el at. (1997b) page 70, subsection "Yield Curve Adjustment" for details.

estimated by the difference between $r^B_{n,t}$ and $r^B_{1,t}$:

$$prem_{n,t} \simeq r^B_{n,t} - r^B_{1,t}. \tag{3.11}$$

Substitute (3.11) into (3.10) to delete the premium term and rearrange the equation, we get

$$1 - \frac{r^B_{n,t} - r^B_{1,t}}{\widetilde{r}_{n,t}} = \frac{1}{n}\sum_{j=0}^{n-1}\left(\frac{\widetilde{r}^e_{1,t+j}}{\widetilde{r}_{n,t}}\right). \tag{3.12}$$

Substitute (3.12) back into (3.9) we obtain

$$\begin{aligned} r^{YCA}_{n,t} &= \frac{1}{n}\sum_{j=0}^{n-1}\left(\frac{\widetilde{r}^e_{1,t+j}}{\widetilde{r}_{n,t}}\right)\widetilde{r}_{n,t} \\ &= \frac{1}{n}\sum_{j=0}^{n-1}\widetilde{r}^e_{1,t+j}, \end{aligned}$$

where $n = 3, 6, 12, 24, 36, 60, t = 1, ..., T - n + 1$. For $t = T - n + 2, T - n + 3, ..., T$, we let

$$\widetilde{r}^e_{1,t} \equiv \widetilde{r}_{1,T-n+1}$$

to calculate $r^{YCA}_{n,t}$, since the interest rate of adjusted demand deposit are sought to be fixed by the central bank. In practice, To approximate this artificial one-month yield, we use adjusted yield on 3-month term deposit times a factor which reflects the difference between 3-month treasury bond yield and 1-month treasury bond yield at the ending period T,

$$\widetilde{r}^e_{1,t} = \widetilde{r}_{3,t}\left(1 - \frac{r^B_{3,T} - r^B_{1,T}}{\widetilde{r}_{3,T}}\right),$$

where $t = 1, ..., T$.

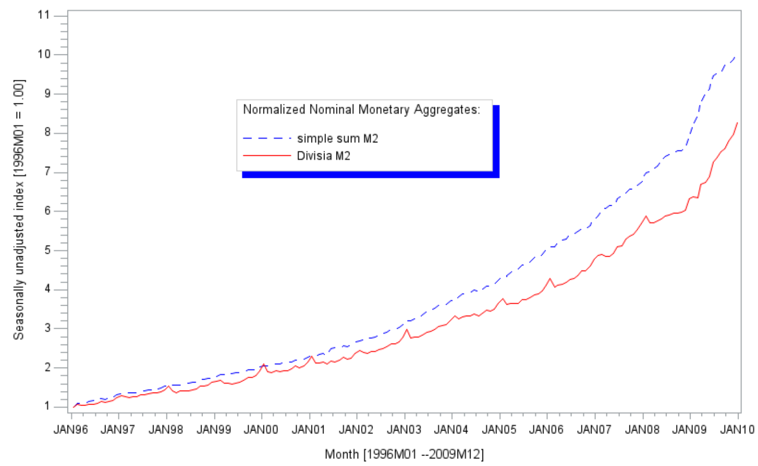

Figure 3.2: Nominal Divisia Quantity Index of Chinese Renminbi M2.

3.5 Divisia index of Chinese Renminbi

We use the nominal balance of monetary assets and compute the real user cost of corresponding monetary assets to construct the nominal Divisia index of Renminbi. We did not follow Yu and Tsui (2000) to construct the real Divisia index because, for different research purposes, different price indexes can be adopted.[12]

Figure 3.2 shows the constructed nominal Divisia M2 and the nominal simple-sum M2 aggregate during January 1996 to December 2009, respectively. Both series are normalized to unity in January 1996, and seasonally unadjusted. We did not seasonal adjust the series because it could destroy important information contained in the series. For further study we suggest first deflate the series by price index which is appropriate for the research purpose, and second include time dummies in the regression.[13] It is clear that Divisia M2 and simple-sum M2 both climbs during the period. However, Divisia M2 is more volatile than Simple-sum M2. It seems that unexpected shocks exist in Divisia M2, but not in Simple-sum M2.

[12]For example, Barnett (1980) used CPI to construct a real Divisia index of the US dollar. Meanwhile, Yu and Tsui (2000) used the Retail Price Index (RPI) as the price deflator to construct real Divisia index of the Chinese Renminbi, see Yu and Tsui (2000), China Economic Review, 11, pp. 139.

[13]Since we constructed monthly Divisia index, we shall include time dummies for each month.

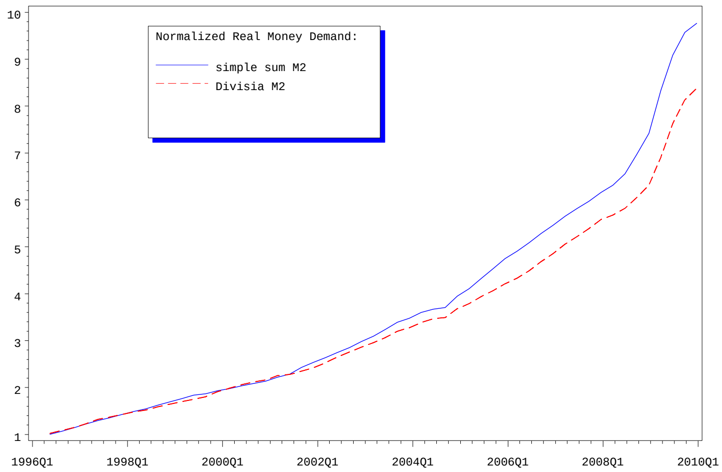

Figure 3.3: Real Divisia Quantity index of Chinese Renminbi M2.

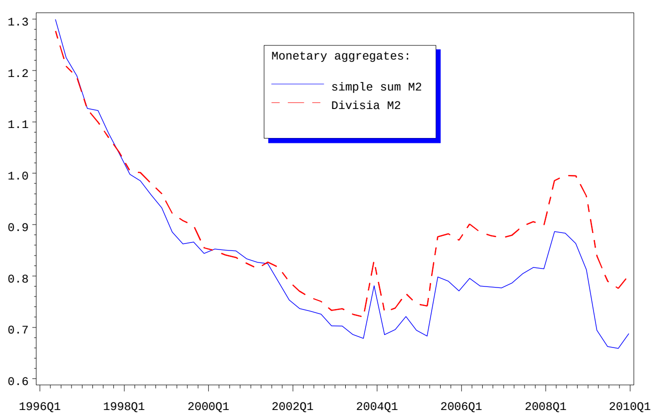

Figure 3.4: Divisia Velocity of Chinese Renminbi M2.

Furthermore, those shocks appear mostly in Januaries or Februaries (but not every year). It could be naively explained by the cultural reason of China – the Spring Festival. Generally, the Spring Festival is in Januaries or Februaries, depends on the traditional Chinese (lunar) calendar. Nevertheless, further investigation seems interesting to be prompted. Table 3.4 summarizes the statistical properties of the Simple-sum M2 and the Divisia M2.

Table 3.4: Statistical Summary of the growth rates of Divisia M2 and Simple-sum M2.

Statistics	Simple-sum M2	Divisia M2
Observations	168	168
Mean	0.013928	0.012512
Median	0.013017	0.013595
Maximum	0.088075	0.102362
Minimum	-0.037566	-0.100885
Standard deviation	0.012872	0.027507
Skewness	1.348449	-0.797074
Kurtosis	8.300271	3.747684

3.6 Conclusion

This essay constructed the Divisia index of Chinese Renminbi. The main contribution of this essay to the literature can be summarized in three aspects. First, we carefully probed the statistical discrepancy of the raw data set provided by the People's Bank of China and proposed the appropriate forms to extract the data of monetary assets' balances. Second, we adjusted the interest rates of China's monetary assets to the annualized one-month holding period yields and further we used the yield curve adjustment method to subtract the term premium of monetary assets with different maturities. Third, we constructed the nominal Divisia index of Renminbi M2 for further research purpose. The constructed Divisia M2 is seasonally unadjusted and seems to contain shocks at the beginning of some years. Further study is triggered because such shocks could be interpreted as structural breaks of money demand function.

Chapter 4

Monetary policy implications of a new-Keynesian DSGE model with theoretically coherent monetary aggregation

By introducing Divisia monetary aggregation into a New Keynesian DSGE model, we showed that the prevailing simple-sum monetary aggregates violated decision optimality conditions. With a continuum of monetary assets and a monetary aggregate, we developed an internally coherent money-demand function, improving understanding of the demand for “moneyness”. The user-cost aggregate, which was dual to the monetary aggregate and was interpreted as the price of “moneyness”, played the key role in our framework and was preferable to the interest rate aggregate or single interest rate most commonly used within such models.

4.1 Introduction

What caused the Great Recession? Many explanations imply irrationality and/or "greed" of economic agents, although the word "greed" is not defined within economic theory. But, in the economic sense, decisions during the period can be treated as rational, given the information sets upon which the decisions conditioned. It has been argued by Barnett and Chauvet (2011a) and Barnett (2011) that a major source of inaccurate information within agents' information sets was the troublesome monetary aggregate data the Fed provided. Those data are inconsistent with elementary principles of aggregation theory over imperfect substitutes. By introducing Divisia monetary aggregation into a New Keynesian DSGE model, we show that the prevailing simple-sum monetary aggregates violate decision optimality conditions and thereby distort decisions. With a continuum of monetary assets and a monetary aggregate, we developed an internally coherent money-demand function, improving understanding of the demand for "moneyness". The user-cost aggregate, which is dual to the monetary aggregate and is interpreted as the price of "moneyness", plays the key role in our framework and is preferable to the interest rate aggregate or single interest rate most commonly used within such models. We propose a monetary policy rule consistent with the model.

4.2 The Model

4.2.1 The Representative Household

We define the *monetary assets* as those assets which provide both liquidity service and store-of-value service. Therefore monetary assets constitute a subset of financial assets, which include those only served as means of store-of-value. The simplest monetary asset is the currency in circulation, which yields no interest. Other monetary asset examples include Negotiable Order of Withdraw (NOW) account, Demand Deposit and Time Deposit. It is obvious that monetary assets yield non-negative interest. Note that if a financial asset provide no liquidity service then it is not counted as monetary asset. For example, a 30-year treasury bond is a financial asset but is not a monetary asset. In our model there exist only one non-monetary financial asset. Since we are considering a riskless world, we shall further call this non-monetary financial asset *risk-free bond,* or *bond* for short. Bond is the *benchmark asset* relative to other monetary assets. Let $M_t(j)$ denotes the nominal balance of monetary asset j in period t. Suppose there exist a continuum of monetary assets represented by the interval $[0, 1]$ with $j \in [0, 1]$.[1] Suppose there exist a *monetary aggregate* M_t, which is a (linear or non-linear) function of individual monetary asset balances:

$$M_t = M_t\left(M_t(j); j \in [0, 1]\right).$$

A representative infinitely-lived household has contemporary utility H_t at period t. We assume that H_t is block-wise weakly separable in current period's consumption of goods and services C_t, holdings of real monetary aggregation M_t/P_t, and hours of labor supplied N_t.[2] Therefore H_t can be written as

$$\begin{aligned} H_t &= H\left(C_t, \frac{M_t}{P_t}, N_t\right) \\ &= H\left(\mathcal{C}(C_t), \mathcal{M}\left(\frac{M_t}{P_t}\right), \mathcal{N}(N_t)\right) \end{aligned}$$

for some monotonically increasing, linear homogeneous, strictly quasi-concave functions $\mathcal{C}(\cdot), \mathcal{M}(\cdot)$ and $\mathcal{N}(\cdot)$.[3] We assume the existence of a continuum of consumption goods and

[1] Actually, the number of monetary assets is finite in practice. But if the number of monetary assets is large enough, it is acceptable to normalize all the monetary assets so that they can be represented by a continuum interval $[0, 1]$.

[2] We use a shortcut that real balance yields utility. Therefore H_t can be viewed as a "derived" utility function from the "true" utility function, which may not depend directly upon real balance of monetary assets. One of such "true" utility function consists of the well known shopping-time model. Croushore (1993) showed that a shopping-time model of money is equivalent to an money-in-utility-function model. See Walsh (2003) Chapter 2 and Barnett and Serletis (2000) pp. 18, footnote 13 for further reference.

[3] The assumption of block-wise strongly seperable utility is very popular in most New Keynesian literature,

services represented by interval $[0,1]$. Consumption index C_t is given by a CES form

$$C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

where $C_t(i)$ represents the quantity of good i consumed by household during period t and $\varepsilon > 0$ governs the elasticity of substitution between two consumption goods. Given goods prices $P_t(i)$ and total expenditure on consumption goods $\int_0^1 P_t(i)C_t(i)di$, household maximize its consumption index C_t so that the optimal allocation of expenditure among consumption goods is

$$C_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} C_t \ , \forall i \in [0,1],$$

where $P_t \equiv \left(\int_0^1 P_t(i)^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}}$ is the price index.

In this framework we assume a perfect competition labor market and flexible wage settings. Bond yields gross rate of return Q_t during period t. For every $j \in [0,1]$, let $R_t(j)$ represents the gross rate of return for monetary asset $M_t(j)$. Then at the beginning of period t, household receives wage income $W_t N_t$, lump-sum transfer T_t, monetary assets interest income $R_{t-1}(j)M_{t-1}(j)$, $\forall j \in [0,1]$ and financial income of matured bond $Q_{t-1}B_{t-1}$ which is purchased at last period. Household allocates its resources into three categories: the purchase of goods and services, the purchase of bond for investment and the purchase of monetary assets for both liquidity and investment. Notice that since bond is a pure investment, $Q_t > R_t(j)$, for every $j \in [0,1]$.

Conditional on current information, the household seeks C_t, M_t/P_t and N_t to maximize its life-time utility

$$E_0 \sum_{t=0}^{\infty} \beta^t H \left(C_t, \frac{M_t}{P_t}, N_t \right).$$

---

e.g.,

$$H_t = H_t \left(\mathcal{C}(C_t) + \mathcal{M} \left(\frac{M_t}{P_t} \right) + \mathcal{N}(N_t) \right)$$

we believe that assumption of strongly seperable utility is partially due to simplification of calculation. Such *ad-hoc* assumption is too simplified and is not desirable in our model.

subject to a sequence of flow budget constraints

$$\int_0^1 P_t(i)C_t(i)di + B_t + \int_0^1 M_t(j)dj = Q_{t-1}B_{t-1} + \int_0^1 R_{t-1}(j)M_{t-1}(j)dj + W_tN_t - T_t$$

for $t = 0, 1, 2, ...$ Household's subjective discount factor β is between 0 and 1. It is important to be clear that rates of return for current period are determined at the beginning of the period, but household can only receive the principal and the interests at the beginning of next period. In fact, by assuming the time interval as $[t, t+1)$, the beginning of period t is included in the time interval $[t, t+1)$, but the ending of period t is not. Therefore, "interest for period t is paid at the beginning of $[t+1, t+2)$" is equivalent to "interest for period t is paid at the end of $[t, t+1)$".

The associated first order conditions can be obtained as

$$\frac{1}{Q_t} = \beta E_t \left\{ \frac{H_{C,t+1}}{H_{C,t}} \frac{P_t}{P_{t+1}} \right\} \tag{4.1}$$

$$\frac{Q_t - R_t(j)}{Q_t} = \frac{H_{M,t}(j)}{H_{C,t}} \text{ , } \forall j \in [0,1] \tag{4.2}$$

$$-\frac{W_t}{P_t} = \frac{H_{N,t}}{H_{C,t}} \tag{4.3}$$

where $H_{C,t} \equiv \partial H_t/\partial C_t$, $H_{M,t}(j) \equiv \partial H_t/\partial(M_t(j)/P_t)$ and $H_{N,t} \equiv \partial H_t/\partial N_t$ represent the marginal utility of consumption, marginal utility of monetary asset $j \in [0,1]$ and marginal utility of labor, respectively. For a more compact notation, let $H_{M,t} \equiv \partial H_t/\partial(M_t/P_t)$. Then using the chained rule,

$$H_{M,t}(j) = \frac{\partial H_t}{\partial(M_t/P_t)} \frac{\partial\left(M_t/P_t\right)}{\partial\left(M_t(j)/P_t\right)} = H_{M,t}\frac{\partial M_t}{\partial M_t(j)}.$$

Let $U_t(j) \equiv (Q_t - R_t(j))/Q_t$. A substitution relationship between consumption and individual monetary asset j can be obtained from household's money demand schedule (4.2):

$$U_t(j)d\left(\frac{M_t(j)}{P_t}\right) = -dC_t.$$

That is to say, if the household considers one unit consumption goods as the numeraire for monetary assets, then the increment of value of one unit j-th monetary asset, $U_t(j)d\left(M_t(j)/P_t\right)$, has to be equal to the increment on the saving of the numeraire, $-dC_t$. Since $d(M_t(j)/P_t)$ is interpreted as the change of the j-th monetary asset's real balance (quantity), $U_t(j)$ can be interpreted as the "price" of the that monetary asset, so that price times quantity would equal to expenditure on the j-th monetary asset. In other words, optimizing household views one unit increase of consumption good indifferent to one unit decrease of real balance of monetary asset j times its discounted foregone interest by holding this monetary asset. When holding monetary asset j, household enjoys its liquidity service but loses the potential interest gained from holding the same amount of bond. This is the well known opportunity cost of holding monetary asset against of holding store-of-value benchmark asset. Specifically, this opportunity cost is given by

$$Q_t - R_t(j).$$

Since the interest is paid at the beginning of next period (or at the end of current period), the household need to discount the "next period's" interest loss to the current period for optimal decision making. Therefore the ratio

$$U_t(j) \equiv \frac{Q_t - R_t(j)}{Q_t}$$

measures the (discounted) opportunity cost of holding monetary asset j. Usually $U_t(j)$ is referred as the *user-cost* or *rental-cost* in literature (e.g. Barnett (1978)).

A semi-weakly separable utility specified in Gali (2008) is relatively simple and sufficient for the purpose of this context:[4]

$$H_t\left(C_t, \frac{M_t}{P_t}, N_t\right) = \frac{X_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}$$

where $0 < \sigma \neq 1$ and $\varphi > 0$. X_t is the aggregation of consumption and real balance with

$$X_t \equiv \left[(1-\theta)C_t^{1-\nu} + \theta\left(\frac{M_t}{P_t}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}}$$

where $\nu > 0$ and $0 < \theta < 1$. Let $i_t \equiv \log Q_t$, $r_t \equiv \log R_t$, $\rho \equiv -\log\beta$, $p_t \equiv \log P_t$ and $\pi_t \equiv p_t - p_{t-1}$. Household's optimal conditions (4.1) and (4.3) can now be rewritten as

$$\frac{1}{Q_t} = \beta E_t\left\{\left(\frac{X_{t+1}}{X_t}\right)^{\nu-\sigma}\left(\frac{C_{t+1}}{C_t}\right)^{-\nu}\frac{P_t}{P_{t+1}}\right\} \tag{4.4}$$

$$\frac{W_t}{P_t} = \frac{N_t^{\varphi}}{(1-\theta)X_t^{\nu-\sigma}C_t^{-\nu}}. \tag{4.5}$$

Equation (4.4) is the household's Euler equation and (4.5) is the household's labor supply schedule. However, without an explicit form of monetary aggregate M_t, we cannot arrive at an analytical function for condition (4.2). Next we will examine the explicit forms for

[4]See Gali (2008), Chapter 2, pp. 27-28. The term "semi-weakly seperable" is in the sense that consumption and real balance are weakly seperable but their aggregation is strongly seperable with labor dis-utility. This assumption is consistent with the two-stage budgeting utility maximization problem. In the first stage, household allocates all its income into two categories: consumption-real balance aggregation X_t and labor supply $\dot{N}_t$:

$$\max H_t = H_t\left(\mathcal{X}\left(X_t\right) + \mathcal{N}(N_t)\right)$$

subject to

$$P_{X,t}X_t + W_tN_t + B_t = Total\ Income$$

where $P_{X,t}$ is the price of consumption-real balance aggregation.

Then in the second stage, household chooses its consumption and real balance given the available resources its allocated in the first stage:

$$\max X_t = X_t(\mathcal{C}(C_t), \mathcal{M}\left(\frac{M_t}{P_t}\right))$$

subject to

$$P_tC_t + U_t\left(M_t/P_t\right) = P_{X,t}X_t$$

where U_t is the user-cost aggregate. We will explicitly explain U_t later. Generally, U_t can be viewed as the "price" of "moneyness" M_t. It can be shown that two-stage budgeting problem is consistent with the utility maximization problem in the main context.

theoretical monetary aggregation in detail.

4.2.2 Theoretical monetary aggregation

Monetary aggregation provided by most central banks are divided in several levels, and it is widely adopted by central banks to calculate higher level of monetary aggregation using simply summation, with unity weight assigned for different monetary assets. For the Federal Reserve, M3 is given by summing Large-denomination Time Deposits, Institutional Money Market Fund Balances, Term Repurchases Agreements and Term Eurodollars to M2.[5] Such algebraically summation is referred as simple-sum (SS) monetary aggregation in literature. However, simple-sum aggregation is not consistent with fundamental economics theories. According to Fisher (1922), the simple arithmetic average–simple-sum aggregation divided by the number of assets to be aggregated–"produces one of the very worst of index numbers", and "should not be used under any circumstances".[6]

To overcome the disadvantage of simple-sum aggregation, literature has been developed for theoretical monetary aggregation (e.g. Barnett (1980)). In this section, we will formally show that simple-sum aggregation violates the consumer's optimal conditions. Furthermore, we shall use a functional form of theoretical aggregation to solve optimal condition (4.2).

Simple-sum monetary aggregation is bad Consider two arbitrary monetary assets j and k from a continuum $[0,1]$. We say that two monetary assets are *different* if and only if for any $j,k \in [0,1]$ and $j \neq k$, $R_t(j) \neq R_t(k)$, where $R_t(j)$ and $R_t(k)$ denote the gross rate of return yielded by holding monetary balance $M_t(j)$ and $M_t(k)$, respectively. Define the *simple-sum monetary aggregation* M_t^{SS} as

$$M_t^{SS} \equiv \int_0^1 M_t(j)dj.$$

[5] Unfortunately M3 is no longer provided by the Federal Reserve.

[6] See Fisher (1922), page 29 and page 361.

Proposition 3 *Simple-sum monetary aggregation M_t^{SS} violates consumer's optimization condition (4.2).*

Proof. (By contradiction.) One can inferred from the definition of Simple-sum aggregation that $\frac{\partial M_t^{SS}}{\partial M_t(j)} = \frac{\partial M_t^{SS}}{\partial M_t(k)} = 1$, $\forall j, k \in [0,1]$. Then for any $j, k \in [0,1]$,

$$\begin{aligned} H_{M^{SS},t}(j) &= H_{M,t}\frac{\partial M_t^{SS}}{\partial M_t(j)} \\ &= H_{M,t}\frac{\partial M_t^{SS}}{\partial M_t(k)} \\ &= H_{M^{SS},t}(k) \end{aligned}$$

Let $\xi_t(j,k) \equiv R_t(j) - R_t(k)$. Then for any two different monetary assets $j, k \in [0,1], j \neq k$, $\xi_t(j,k) \neq 0$. Thus,

$$\begin{aligned} \frac{H_{M^{SS},t}(j)}{H_{c,t}} &= \frac{H_{M^{SS},t}(k)}{H_{c,t}} \\ &= \frac{Q_t - R_t(k)}{Q_t} \\ &= \frac{Q_t - R_t(j)}{Q_t} + \frac{\xi_t(j,k)}{Q_t} \\ &\neq \frac{Q_t - R_t(j)}{Q_t} \end{aligned}$$

contradict with consumer's optimization condition (4.2). ■

Proposition 3 shows that the central banks' decisions based on simple-sum monetary aggregation M_t^{SS} is inconsistent with economics optimality conditions. This proposition is discussed literally in Barnett and Chauvet (2011). In the same paper, the authors also showed that when focusing on the money supply using simple-sum monetary aggregation, the Federal Reserve is possibly responsible for the early 1980's recession as well as other recessions. Proposition 3 confirms their findings and provides a mathematical foundation to support those empirical evidence.

Theoretical Monetary Aggregation If the widely used simple-sum monetary aggregate cannot fulfill its job, then what is the "right" monetary aggregate that economists should use? We follow Belongia and Ireland (2010) to define theoretical monetary aggregation M_t takes the following functional form:

$$M_t \equiv \left(\int_0^1 \eta(j)^{\frac{1}{\omega}} M_t(j)^{\frac{\omega-1}{\omega}} dj \right)^{\frac{\omega}{\omega-1}}. \tag{4.6}$$

Parameter $\eta(j)$ governs steady-state expenditure share on monetary asset $M_t(j)$ and satisfies $0 < \eta(j) < 1$ with $\int_0^1 \eta(j)dj \equiv 1$. Parameter ω governs the elasticity of substitution between different monetary assets. Belongia and Ireland (2010) showed that simple-sum monetary aggregation M_t^{SS} and theoretical monetary aggregation M_t exhibit different impulse responses given monetary policy shock or preference shock under conventional parameter calibrations.

Now household optimal condition (4.2) can be rewritten as

$$U_t\left(j\right) = \frac{\theta \left(\frac{M_t}{P_t}\right)^{-\nu} \left[\eta(j)\frac{M_t}{M_t(j)}\right]^{\frac{1}{w}}}{(1-\theta)C_t^{-\nu}}, \forall j \in [0,1] \tag{4.7}$$

Equation (4.7) can be viewed as a money demand function for monetary asset j. Since we have infinite number of monetary assets, we shall have infinite number of individual money demand functions. Dealing with so many money demand functions requires prior knowledge of the structure of interest rates, e.g., how the difference of two difference monetary assets' interest rates $\xi_t(j,k) \equiv R_t(j) - R_t(k)$ is determined. Our strategy is to derive an internally coherent money demand function to avoid assuming structural form of $\xi_t(j,k)$. Define the *user-cost aggregate* U_t which is dual to monetary aggregate M_t

$$U_t \equiv \left(\int_0^1 \eta(j) U_t(j)^{1-\omega} dj \right)^{\frac{1}{1-\omega}}. \tag{4.8}$$

It can be shown that $\int_0^1 U_t(j)M_t(j) = U_t M_t$. The following proposition states the functional form of the coherent money demand function.

Proposition 4 *Given theoretical monetary aggregate (4.6) and its dual (4.8), the internally coherent money-demand function can be obtained as*

$$U_t = \left(\frac{\theta}{1-\theta}\right)\left(\frac{C_t}{M_t/P_t}\right)^{\nu}. \tag{4.9}$$

Proof. *See Appendix.* ■

Maybe it is more obvious to view (4.9) in its log-linearized form

$$m_t - p_t = c_t - \frac{1}{\nu}u_t$$

which is up to an additive constant term $\frac{1}{\nu}\left(\log\frac{\theta}{1-\theta}\right)$ on the right hand side. Equation (4.9) is with the same structural form of canonical money demand function. However, (4.9) has a different interpretation: it shows the household's demand of "moneyness" M_t, instead the demand of any particular monetary asset $M_t(j)$. If currency in circulation is the only monetary asset, its demand function with the benchmark asset interest rate Q_t is sufficient for both theoretical and practical purpose. But if there are more than two monetary assets, with at least one of them yields positive interest, it is difficult to argue which monetary asset is ought–and how–to be included in "the" money demand. Equation (4.9) circumvent that embarrassment by providing an aggregation level money demand function. As $\nu > 0$, one can interpret the above equation as: if the user-cost of monetary aggregate–the opportunity cost of using "moneyness"–goes up, it is more expensive to hold monetary aggregate, so household shall reduce its demand of monetary aggregate.

Denote lower case letter for logarithm of upper case letter, e.g., $z_t \equiv \log Z_t$ for any generic variable Z_t. Log-linearize optimal conditions (4.4), (4.9) and (4.5) around a perfect foresight steady state with constant inflation, constant consumption growth rate and constant user-cost growth rate yields[7]

$$
\begin{aligned}
i_t - E_t\{\pi_{t+1}\} - \rho &= \sigma E_t\{\Delta c_{t+1}\} + \frac{\lambda}{\nu} E_t\{\Delta u_{t+1}\} && (4.10)\\
m_t - p_t &= c_t - \frac{1}{\nu} u_t && (4.11)\\
w_t - p_t &= \sigma c_t + \varphi n_t + \frac{\lambda}{\nu} u_t && (4.12)
\end{aligned}
$$

where $\lambda \equiv \chi(\nu - \sigma)$ with $\chi \equiv \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{\nu}} U^{\frac{1-\nu}{\nu}}\right]^{-1}$ where U stands for user-cost index at the steady state. Notice that $0 < \chi < 1$. In fact, define steady state velocity $V \equiv \frac{C}{M/P}$ and use equation (4.9) at its steady state, we shall have $V \equiv \left(\frac{1-\theta}{\theta} U\right)^{\frac{1}{\nu}}$. Therefore $\chi = \frac{U}{V+U} \in [0, 1]$.

User-cost aggregate u_t plays the key role in this framework: it enters Euler equation (4.10), determines demand of monetary aggregate by (4.11) and household labor supply schedule by (4.12). Therefore household's optimal choices of marginal utility of consumption, marginal utility of monetary assets and marginal utility of labor are linked by one channel: the user-cost aggregate u_t.

4.2.3 Firms

Assume a continuum of firms indexed by $i \in [0, 1]$. Each firm produce different product but use identical technology. The product function of firm i is given by

$$Y_t(i) = A_t N_t(i)^{1-\alpha} \tag{4.13}$$

where A_t represents the level of technology and $N_t(i)$ stands for the labor demanded by firm i. Firms take aggregate price index P_t and aggregate consumption index C_t as given. We assume that the technology progress is exogenous. Specifically,

$$a_t = \phi_a a_{t-1} + \varepsilon_{a,t} \tag{4.14}$$

[7] These log-linearized first order conditions are up to additive constant terms. However, we ignore the constants since they only affect variables levels but not variables dynamics (impulse responses).

where $a_t \equiv \log A_t$. Serial correlation ϕ_a is between 0 and 1. Stochastic shock $\varepsilon_{a,t}$ is *i.i.d.* with mean zero and variance σ_a^2.

Goods market is monopolistic comparative and firms price their products according to Calvo (1983). Specifically, for any period, a firm may reset its optimal price with probability $1-\gamma$ which is independent of time. Thus during any period, a fraction of $1-\gamma$ of all firms reset their prices and a fraction of γ keep their prices unchanged. We shall skip the derivations of price dynamics and firms marginal cost to keep the context short.

4.2.4 Equilibrium

Goods market clear requires that, at each period and for every good in the market, household good demand is equal to firms good supply,

$$Y_t(i) = C_t(i)$$

for all $i \in [0,1]$ and $t = 0, 1, 2, ...$ Define the aggregate output to be $Y_t \equiv \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}$, it turns that

$$Y_t = C_t.$$

Combining the goods market equilibrium condition with the consumer's Euler equation (4.10), we will arrive at

$$i_t - E_t\{\pi_{t+1}\} - \rho = \sigma E_t\left\{\Delta y_{t+1}\right\} + \frac{\lambda}{\nu} E_t\left\{\Delta u_{t+1}\right\}.$$

Furthermore, labor market clear requires that the representative household labor supply fulfills the aggregation of firm labor demand,

$$N_t = \int_0^1 N_t(i) di.$$

Combining with firms production plan (4.13), and take log-linearization, we may have the aggregated employment

$$n_t = \frac{1}{1-\alpha}\left(y_t - a_t\right)$$

which states that the economy's employment is determined by the national output and technology level.

Let y_t^n denotes the natural output, which is defined as the equilibrium level of output under flexible prices. It can be obtained that in our model setup the natural output is determined by technology and real balance of monetary aggregate, therefore y_t^n is no longer exogenous. Specifically,[8]

$$\Delta y_t^n = \tau_a \Delta a_t + \tau_m \Delta m_t \tag{4.15}$$

where

$$\begin{aligned} \tau_a &\equiv \frac{\frac{1+\varphi}{1-\alpha}}{(\sigma+\lambda)+\frac{\varphi+\alpha}{1-\alpha}} \\ \tau_m &\equiv \frac{\lambda}{(\sigma+\lambda)+\frac{\varphi+\alpha}{1-\alpha}}. \end{aligned}$$

We are interested in the signs of τ_a and τ_m. Using the fact that $\lambda \equiv \chi(\nu-\sigma)$, we can obtain that for the common denominator

$$(\sigma+\lambda)+\frac{\varphi+\alpha}{1-\alpha} = \chi\nu + (1-\chi)\,\sigma + \frac{\varphi+\alpha}{1-\alpha} > 0,$$

since $0 < \chi < 1$. Therefore $\tau_a > 0$. For τ_m, the denominator is positive, so the sign of τ_m is determined by $\lambda \equiv \chi(\nu-\sigma)$ which in turn by $(\nu-\sigma)$, the difference between the elasticity of substitution for intra-temporal, ν, and the elasticity of substitution for inter-temporal, σ. Please note that ν is also the reciprocal semi-elasticity of output with respect

[8] As we shall show later, the money supply growth rate Δm_t can be determined by the central bank. Therefore, the monetary authority would have the capability to affect not only the actual output, but also the natural output. However, in the later context we assumed that the central bank is targeting on output gap, which is defined as the actual output minus the natural output.

to the aggregated user-cost according to (4.11). If $\nu > \sigma$, or elasticity of substitution for intra-temporal is greater than for inter-temporal, then the real balance has a positive effect on natural output, $\tau_m > 0$. Otherwise if $\nu < \sigma$, then real balance has a negative effect on natural output, $\tau_m > 0$. A special case is that when $\nu = \sigma$, we will have $\tau_m = 0$ and potential output is immune from the real balance.[9]

It is well developed in literature that under this type of framework, we can obtain two conditions. Define $\widehat{y}_t \equiv y_t - y_t^n$ to be the output gap, one condition is the New Keynesian Philips curve (NKPC):

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \kappa \widehat{y}_t \tag{4.16}$$

where $\kappa \equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}(1-\beta\theta)\left(\frac{1-\theta}{\theta}\right)\left(\sigma + \lambda + \frac{\alpha+\varphi}{1-\alpha}\right) > 0$.

The other condition is the dynamic IS (DIS) curve:

$$\widehat{y}_t = E_t\{\widehat{y}_{t+1}\} - \frac{1}{\sigma+\lambda}\left[i_t - \rho - (1+\lambda)E_t\{\pi_{t+1}\} + \lambda E_t\{\Delta m_{t+1}\}\right] + E_t\{\Delta y_{t+1}^n\}. \tag{4.17}$$

Two points should be clarified before we go further. First, the user-cost index growth rate Δu_t is a factor to determine output gap $\widehat{y}_t$, which means monetary aggregate is non-neutral. This is a result of block-wise weakly separable utility function. If household utility is strongly separable, then only the bond rate of return, which can be expressed as i_t, will appear in DIS. That means neutrality of monetary aggregate, even if we are in a New Keynesian environment.

Second, more importantly, the bond interest rate i_t in (4.17) is not the traditional interest

[9] Since we have assumed a strong seperable utility function between the aggregated consumption-real balance and the labor supply, the inter-temporal elasticity of substitution w.r.t aggregated consumption-real balance can be obtained as

$$-\frac{\partial \ln\left(\frac{\partial H_{t+1}/\partial X_{t+1}}{\partial H_t/\partial X_t}\right)}{\partial \ln\left(X_{t+1}/X_t\right)} = \sigma$$

and the intra-temporal elasticity of substitution between consumption and real balance can be obtained as

$$-\frac{\partial \ln\left(\frac{\partial (M_t/P_t)}{\partial C_t}\right)}{\partial \ln\left(\frac{M_t/P_t}{C_t}\right)} = \nu$$

rate to be controlled by any type of policy rules. This is obvious since in our framework, benchmark asset, or the bond, is a pure investment and provides full service of store-of-value. Empirically, Anderson et. al. (1997) identified the benchmark rate as the "envelope" of the each individual monetary asset rate of return and the rate on Moody's seasoned BAA bonds, $R_{BAA,t}$

$$Q_t \equiv \max\left\{R_{BAA,t},\ R_t(j);\ \forall j \in [0,1]\right\}$$

with $i_t = \log Q_t$. For simplicity, suppose Moody's BAA bond rate of return is always the maxima. Then it is non-sensible to assume that central bank has a direct capability to control financial market return. In fact, central bank controls the aggregate user-cost u_t (or monetary assets aggregate m_t), which will then enter the household's Euler equation for inter-temporal substitution of consumption. When household is determining its current period consumption and next period consumption, it faces (1) *given* monetary assets aggregate use-cost u_t, which is controlled by central bank, (2) benchmark asset rate of return i_t, which is determined exogenously by the financial market and (3) the utility function which is revealed to itself.

We assume there is a market maker who provides the bond to household according to an approximation of household's Euler equation.[10] In specific, the bond rate is determined by the expected output growth rate, the expected inflation rate, and the expected aggregated monetary asset growth rate

$$\widetilde{i}_t = \phi_y E_t\{\Delta y_{t+1}\} - \phi_m E_t\{\Delta m_{t+1}\} + \phi_\pi E_t\{\pi_{t+1}\} + \varepsilon_{i,t} \qquad (4.18)$$

for $t = 1, 2, 3, ...$ and $\widetilde{i}_t \equiv i_t - \rho$. Notice that $\rho \equiv -\log \beta$ is the steady state benchmark asset rate of return. This is implied by the steady state level of (4.17). $\varepsilon_{i,t}$ is an *iid* random shock

[10] Another way to view this approximation is that benchmark bond rate i_t is determined by forward looking Arbitrage Pricing Theory (APT) type process. It is easy to show that the household's Euler equation can be rewritten as

$$\widetilde{i}_t = (\sigma + \lambda)\, E_t\{\Delta y_{t+1}\} - \lambda E_t\{\Delta m_{t+1}\} + (1 + \lambda)\, E_t\{\pi_{t+1}\}$$

. Therefore the well known APT model can be viewed as another form of Euler equation of the DSGE model.

with zero mean and standard deviation σ_i. Based empirical evidence of financial economics, we assume the process coefficients ϕ_y, ϕ_m, and ϕ_π all within the range of unit circle. Given equation (4.18), (4.16) and (4.17), the model will be closed by providing monetary policy rules for aggregate user-cost index u_t.

4.3 Monetary policy under theoretical monetary aggregation

We first focus on specifying a plausible monetary policy rule coherent with our model setup. To be directly comparable to existing New Keynesian literature, it is possible to specify a Taylor-type rule which targets on interest rate–and in our context, the user-cost aggregate u_t.[11] Although the Taylor-type rule is popular, we are reluctant to follow literature in that way. Our arguments are twofold: First, as Adam and Billi (2005) pointed out, there exists a zero lower bound (ZLB) problem with the Taylor-type rule and thus makes theoretical framework non-linear. Possible answer to solve this problem is simple: we can target on monetary aggregate or its growth rate. For an extreme case, when interest rate of monetary assets or the Federal Funds rate is zero, the monetary authority can always expend money supply. Thus it seems that targeting on money supply is a superior choice when the market interest rate is zero or near zero. Practical experience has also shown that during a low-interest-rate era, the monetary authority prefers targeting on money supply. Examples include the "quantitative easing" used by Bank of Japan during 1990's and by the Federal Reserve during the Great Recession.

Second, instead of the fact that Federal Funds rate is directly observable, the user-cost aggregate u_t is not. The disadvantage for central bank to control the theoretical user-cost aggregate is that u_t depends upon functional form (4.8), which is hardly known in practice.

[11] Specifically,

$$u_t = \log U + \phi_\pi^u \pi_t + \phi_y^u \widetilde{y}_t$$

where $\widetilde{y}_t \equiv y_t - y$ is the (log) deviation of output from its steady state.

To overcome the second disadvantage, index numbers are introduced into economics literature. Christensen et al. (1971) showed that a homogeneous trans-log quandratic polynomial is capable to provide a second-order approximation to an arbitrary twice differentiable linear homogeneous function. In addition, Diewert (1976) defined a class of exact and superlative index numbers which included Divisia index. Based on Diewert's work, Barnett (1980) has shown that Divisia price index or quantity index is independent on any functional forms of M_t. In that paper, the author defined the "user-cost" of monetary assets as

$$U_t(j) \equiv \frac{Q_t - R_t(j)}{Q_t}$$

for all $j \in [0,1]$, which is exactly the same as we did in this framework. Barnett (1980) also showed that Fisher's Ideal index and Törnqvist-Theil "Divisia" index are consistent with the consumer's optimal decisions. The two index numbers give almost identical results, but Fisher's Ideal index is more complicated than Törnqvist-Theil "Divisia" index to compute. Consequently literature adopted Törnqvist-Theil "Divisia" index. We shall further call it Divisia index for convenience. The following definition formally specifies Divisia quantity and user-cost index. Recently, Barnett and Chauvet (2011) graphically showed that before almost every recession of the US during the recent 50 years, there was an dramatic (but possibly unintended) decrease of the money supply measured by Divisia monetary index.

Definition 5 *For all monetary assets $j \in [0,1]$, and given monetary asset balance $M_t(j)$ and its user-cost $U_t(j)$, Divisia quantity index M_t^D is defined as*

$$\log M_t^D - \log M_{t-1}^D = \int_0^1 \overline{s}_t(j)\left[\log M_t(j) - \log M_{t-1}(j)\right] dj \tag{4.19}$$

and Divisia user-cost index U_t^D is defined as

$$\log U_t^D - \log U_{t-1}^D = \int_0^1 \overline{s}_t(j)\left[\log U_t(j) - \log U_{t-1}(j)\right] dj \tag{4.20}$$

where

$$\overline{s}_t(j) \equiv \frac{s_t(j) + s_{t-1}(j)}{2}$$

with

$$s_t(j) \equiv \frac{U_t(j)M_t(j)}{\int_0^1 U_t(k)M_t(k)dk}$$

represents the expenditure share of monetary asset $M_t(j)$ among the total expenditure on monetary assets.

One can observe that Divisia quantity index of monetary aggregation is given by its growth rate. For a more compact formation, let

$$\Delta m_t^D = \log M_t^D - \log M_{t-1}^D$$

then equation (4.19) can be expressed as

$$\Delta m_t^D = \int_0^1 \overline{s}_t(j)\Delta m_t^D(j)dj$$

where $\Delta m_t^D(j) = \log M_t(j) - \log M_{t-1}(j)$ is the growth rate for an arbitrary monetary asset.

The merit of introducing index number theory into aggregation theory is to non-parametrically approximate the unknown structure of monetary aggregation so that economists or econometricians are neither forced to assume any structure of monetary aggregation nor to estimate (or calibrate) associated parameters. In fact, Belongia and Ireland (2010) has confirmed that theoretical monetary aggregation with a form of (4.6) and Divisia monetary index (4.19) exhibits almost identical impulse responses under a variety of shocks. The growth rate of Divisia quantity index Δm_t^D is directly computable from available data. Therefore central bank would be no more difficult to target on Δm_t^D than on the Federal Funds rate.

4.4 The Dynamic System of Economy

Consider a monetary policy rule targeting on monetary aggregate growth rate:

$$\Delta m_t = \varphi_{y0}\widehat{y}_t + \varphi_{y_1}\widehat{y}_{t-1} + \varphi_\pi \pi_t + \zeta_t \tag{4.21}$$

where φ_π should be non-positive. Monetary policy shock ζ_t is a first order auto-regressive stochastic process. In specific,

$$\zeta_t = \phi_\zeta \zeta_{t-1} + \varepsilon_{\zeta,t} \tag{4.22}$$

for $t = 1, 2, 3, ...$ and $\phi_\zeta \in [0, 1)$.

The economy's forward-looking dynamic system of NKPC (4.16), DIS (4.17), and policy rule (4.21) can be written more compactly as a system of two-variable, simultaneous equations

$$\begin{pmatrix} \widehat{y}_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} \frac{B}{A} & \frac{C}{A} \\ \kappa\frac{B}{A} & \beta + \kappa\frac{C}{A} \end{pmatrix} \begin{pmatrix} E_t\left\{\widehat{y}_{t+1}\right\} \\ E_t\left\{\pi_{t+1}\right\} \end{pmatrix} + S_t \begin{pmatrix} \frac{1}{A} \\ \frac{\kappa}{A} \end{pmatrix}.$$

where

$$\begin{aligned}
A &= 1 - \left(\tau_m - \frac{\lambda - \phi_m}{\sigma + \lambda - \phi_y}\right)\varphi_{y_1} \\
B &= 1 + \left(\tau_m - \frac{\lambda - \phi_m}{\sigma + \lambda - \phi_y}\right)\varphi_{y_0} \\
C &= \frac{1 + \lambda - \phi_\pi}{\sigma + \lambda - \phi_y} + \left(\tau_m - \frac{\lambda - \phi_m}{\sigma + \lambda - \phi_y}\right)\varphi_\pi \\
S_t &= \left(\tau_m - \frac{\lambda - \phi_m}{\sigma + \lambda - \phi_y}\right)\phi_\zeta \zeta_t + \tau_a(\phi_a - 1)a_t - \frac{1}{\sigma + \lambda - \phi_y}\varepsilon_{i,t}
\end{aligned}$$

with

$$
\begin{aligned}
\kappa &\equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}(1-\beta\theta)\left(\frac{1-\theta}{\theta}\right)\left(\sigma+\lambda+\frac{\alpha+\varphi}{1-\alpha}\right)>0 \\
\tau_a &\equiv \frac{\frac{1+\varphi}{1-\alpha}}{(\sigma+\lambda)+\frac{\varphi+\alpha}{1-\alpha}}>0 \\
\tau_m &\equiv \frac{\lambda}{(\sigma+\lambda)+\frac{\varphi+\alpha}{1-\alpha}}.
\end{aligned}
$$

The components of S_t shows that the shock terms contains three components, exogenous technology a_t, exogenous benchmark asset rate of return shock $\varepsilon_{i,t}$, and monetary policy shock ζ_t which is determined by the central bank.

The above dynamic system has two non-determined variables, $\widehat{y}_t$ and π_t. To this type of system, Blanchard and Kahn (1980) have shown that the solution is locally unique if and only if the coefficient matrix

$$
\begin{pmatrix} \frac{B}{A} & \frac{C}{A} \\ \kappa\frac{B}{A} & \beta+\kappa\frac{C}{A} \end{pmatrix}
$$

has both eigenvalues within the unit circle. Write out the associated characteristic polynomial

$$
p(\xi)=\xi^2-\left(\frac{B}{A}+\beta+\kappa\frac{C}{A}\right)\xi+\beta\frac{B}{A} \tag{4.23}
$$

Proposition 6 *Characteristic polynomial (4.23) has both of its roots with the unit circle if and only if*

$$
\begin{aligned}
\left|\beta\frac{B}{A}\right| &< 1 \qquad\qquad (4.24)\\
\left|\frac{B}{A}+\beta+\kappa\frac{C}{A}\right| &< 1+\beta\frac{B}{A}.
\end{aligned}
$$

Proof. See LeSalle (1986, p. 28). ■

In practice, we did a numerical simulation on inequalities (4.24). Using the calibrated parameters that would be shown later, we found a combination of the parameter set that

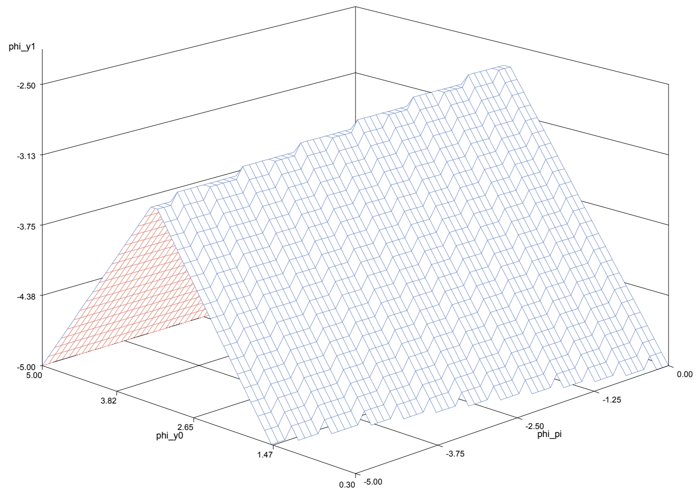

Figure 4.1: Determinacy Surface of Monetary Policy Rule

satisfied inequalities (4.24). All area below the illustrated surface of Figure 4.1 are the feasible parameter combination that satisfy inequalities (4.24).

4.4.1 Calibration

We consider the Moody's Baa bond as the benchmark asset of financial market rate. Household's subjective discount factor is assumed to be $\beta = 0.9876$, implying that $\rho = -\log \beta = 0.01248$, which states that in its steady state the Moody's Baa rate of return is roughly 5% annually. We assume a unitary elasticity of labor supply, so $\varphi = 1$. The setting of $\theta = 1/3$ means household treat consumption and real balance equally. The steady state user-cost aggregate is set $U = 0.5$.

For the firms side, we set $\alpha = 1/3$ meaning that labor elasticity of marginal production is approximately -0.3. It is assumed $\varepsilon = 6$ which implies average frictionless mark-up to be 20%. In addition $\gamma = 2/3$ implies average price rigidity period of 3 quarters.

For the shock components, we set technology process (4.14) $\phi_a = 0.9$, which means that technology shock is highly persistent. For the financial market rate of return process (4.18), we set $\phi_y = 0.9$, $\phi_m = 0.9$, and $\phi_\pi = 0.9$. For the monetary policy shock process (4.22), we set $\phi_\zeta = 0.8$. Since variances of shocks only affect the magnitute of the impulse response functions, we set $\sigma_a = \sigma_i = 0.1$.

We have left out the calibration for inter-temporal elasticity of substitution, σ, intra-temporal elasticity of substitution, ν, and monetary authority chosen parameters φ_{y_0}, φ_{y_1}, and φ_π. The combination of these parameter calibrations shall satisfy inequalities (4.24). Considering the complexity of our system and the non-linearity of the inequalities (4.24), we applied numerical analysis to the parameter values instead of analytically solve the inequalities.

4.4.2 Impulse Responses

In our framework, monetary aggregate enters the determination of natural output. Remind that

$$\Delta y_t^n = \tau_a \Delta a_t + \tau_m \Delta m_t$$

where

$$\begin{aligned} \tau_a &\equiv \frac{\frac{1+\varphi}{1-\alpha}}{(\sigma+\lambda)+\frac{\varphi+\alpha}{1-\alpha}} \\ \tau_m &\equiv \frac{\lambda}{(\sigma+\lambda)+\frac{\varphi+\alpha}{1-\alpha}}. \end{aligned}$$

We examined the impulse responses when $\nu > \sigma$, and therefore $\tau_m > 0$. For further calibrating and deriving the impulse response functions, we set the intertemporal elasticity of substitution $\sigma = 0.2$. The intra-temporal elasticity of substitution is assumed to be $\nu = 0.8$, which also implies that the semi-elasticity of the user-cost aggregate is 2 by the aggregated money demand schedule (4.11). Central bank's reaction to inflation is set $\varphi_\pi = -0.6$, and reaction to output growth rate are set $\varphi_{y_0} = 3.1$ and $\varphi_{y_1} = -4.9$ to satisfy the inequalities (4.24) implied by eigenvalue characteristic polynomials.

Output gap and inflation exhibit similar curveture as responses to monetary policy shock, despite the magnitude. That's because we have a linear dynamic system. However, employ-

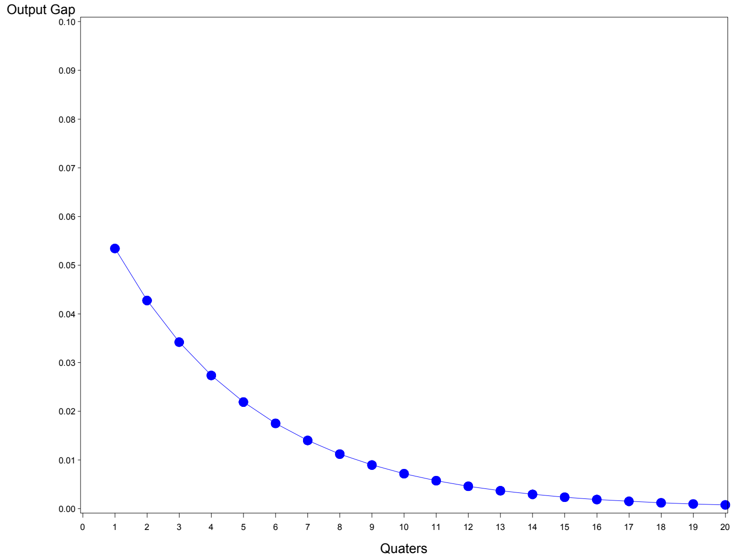

Figure 4.2: Output Gap to Monetary Policy Shock

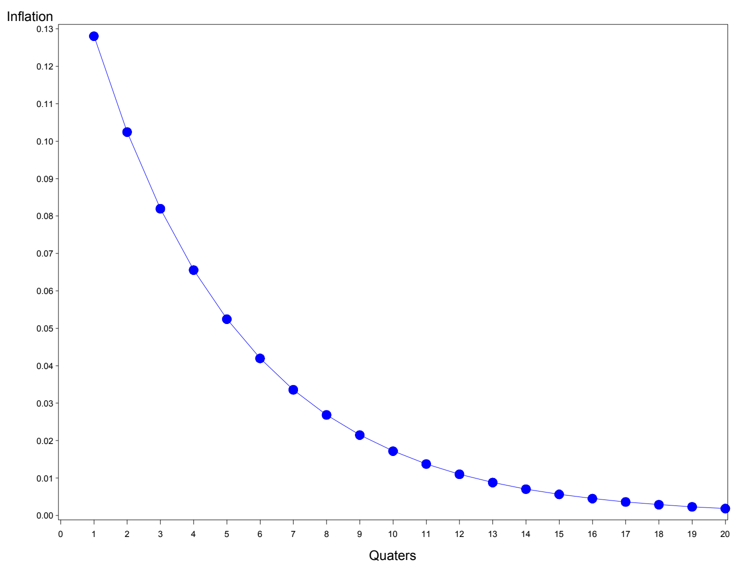

Figure 4.3: Inflation to Monetary Policy Shock

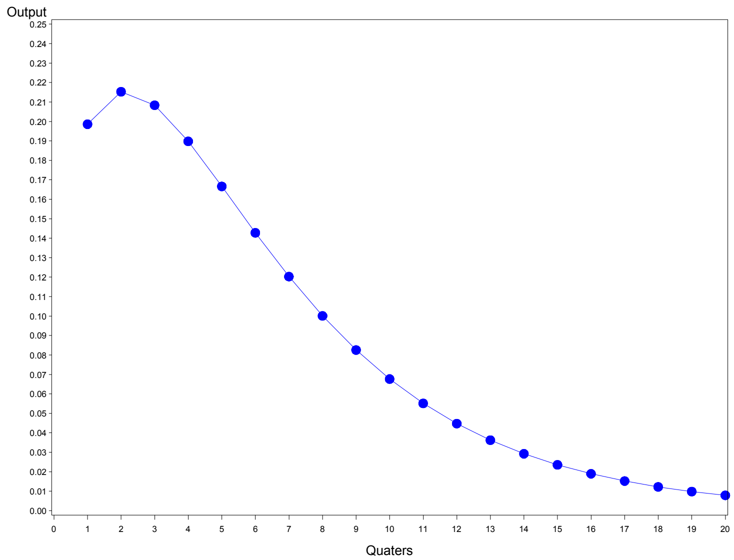

Figure 4.4: Output to Monetary Policy Shock

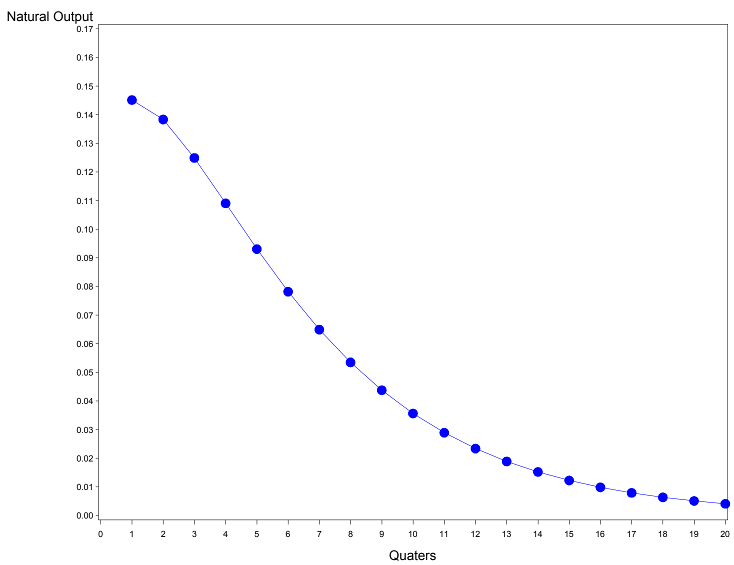

Figure 4.5: Natural Output to Monetary Policy Shock

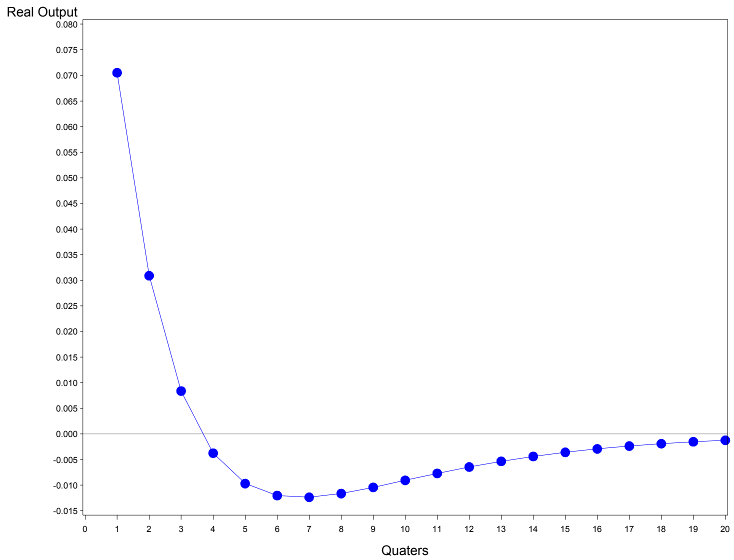

Figure 4.6: Real Output to Monetary Policy Shock

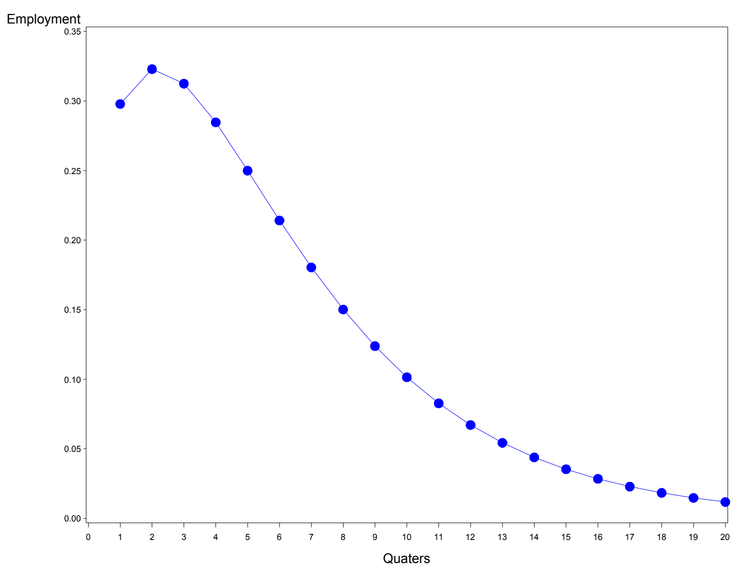

Figure 4.7: Employment to Monetary Policy Shock

ment, output, and real output responses exhibit different curveture comparing with the output gap responses. Since in our framework, monetary aggregate affects the natural output, we will obtain an impulse response of natural output given monetary policy shock.

Table 4.1: Parameters Description and Calibration

Parameter	Calibration	Description And Implication
β	0.9876	*Household Discount Factor*
ρ	0.01248	*Steady State benchmark rate.* $\rho = log(\beta)$
σ	0.2	*Household inter-temporal elasticity of substitution*
φ	1	*Household Elasticity of Labor Supply*
ν	0.8	*Household intra-temporal elasticity of substitution. Implies semi-elasticity of benchmark rate is* 1.25
θ	1/3	*Household treat consumption and real balance equally*
U	0.5	*Steady state user-cost aggregate*
α	1/3	*Implies labor elasticity of marginal product is* -0.3
ε	6	*Implies average frictionless mark-up as* 0.2
γ	2/3	*Implies average price rigidity of 3 quarters*
φ_π	-0.6	*Central bank response to inflation*
φ_{y_0}	3.1	*Central bank response to current period output gap*
φ_{y_1}	-4.9	*Central bank response to last period output gap*
ϕ_a	0.9	*First order auto-correlation of technology*
ϕ_ζ	0.8	*First order auto-correlation of monetary aggregate*
σ_a	0.01	*Variance of technology shock*
σ_i	0.01	*Variance of financial institutional shock*
σ_ζ	0.01	*Variance of monetary aggregate shock*
ϕ_y	0.5	*Market interest rate response to expected output growth rate*
ϕ_m	0.18	*Market interest rate response to expected monetary aggregation growth rate*
ϕ_π	1.3	*Market interest rate response to inflation rate*

Chapter 5

Concluding Remarks

In section 2, The Idiosyncratic Dynamics of the Real Exchange Rates, I used a dynamic factor model to isolate the real exchange rates idiosyncratic dynamics from the contamination of the numeraire country's unexpected shock. I then proposed a new estimation of the common feature to capture the numeraire shock. The convergence of idiosyncratic dynamics of real exchange rates was tested and converging speed was estimated by adopting a Two Stage Least Square method. We found that under the common feature, the evidence of real exchange rates convergence for 20 OECD countries from 1973 to 1998 was much weaker than previous literature documented. And also, the convergence of real exchange rates was not robust to priori parameter settings of lag length selection combining with size control of serial correlation. However, the evidence of idiosyncratic real exchange rates convergence was stronger in the 1990's. For some cases of priori parameter specifications, the point estimation of half-life could be 18 months with a very tight confidence interval below three years.

In section 3, Constructing the theoretical monetary aggregation for Chinese Renminbi, was devoted into constructing of the accurate monetary aggregation of China's RMB. We reviewed Barnett (1980)'s Divisia monetary index theory, discussed the adjustment of interest rates for China's own case and constructed the Divisia index of Chinese Renminbi. The rapid economic growth and quick financial innovations implies potential structural changes of China's monetary demand, thus the up-to-date version of Divisia index of RMB is needed for further study purpose.

Section 4, Monetary policy implications of a new-Keynesian DSGE model with theoretically coherent monetary aggregation, showed a New Keynesian framework including theoretical monetary aggregation in household's utility function. It has been argued by Barnett and Chauvet (2011a) and Barnett (2011) that a major source of inaccurate information within agents' information sets was the troublesome monetary aggregate data the Fed provided. Those data were inconsistent with elementary principles of aggregation theory over imperfect substitutes. By introducing Divisia monetary aggregation into a New Keynesian DSGE model, we showed that the prevailing simple-sum monetary aggregates violated decision optimality conditions and thereby distorted decisions. With a continuum of monetary assets and a monetary aggregate, we developed an internally coherent money-demand function, improving understanding of the demand for "moneyness". The user-cost aggregate, which was dual to the monetary aggregate and was interpreted as the price of "moneyness", played the key role in our framework and was preferable to the interest rate aggregate or single interest rate most commonly used within such models. We proposed a monetary policy rule consistent with the model, and studied the impulse response to monetary policy shock.

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Appendix A

Supplement to Chapter 2

A.1 Revisit of estimation procedure

In this appendix we revisit LLC's Two Stage Least Square estimation technique. Consider the empirical model (2.10). At the first stage, after determine auto-regression lag order p_i by General-to-specific identification method, we could run tow auxiliary regressions. That is, regress Δe_{it} and $e_{i,t-1}$ on the lagged difference terms $\Delta e_{i,t-j}$ through $i = 1, ..., p_i$, respectively, to produce orthogonal residuals. Save the residuals from each of the auxiliary regressions. In detail, those two pre-regressions are given by

$$\Delta e_{it} = \sum_{j=1}^{p_i} \hat{\psi}_{ij} \Delta e_{i,t-j} + \hat{u}_{it}$$

and

$$e_{it-1} = \sum_{j=1}^{p_i} \hat{\phi}_{ij} \Delta e_{i,t-j} + \hat{v}_{it-1}$$

After obtaining the two orthogonalized residuals $\hat{u}_{it}$ and $\hat{v}_{it-1}$, and before the second stage of regression, we need to normalize the residuals $\hat{u}_{it}$ and $\hat{v}_{it-1}$ to control for heterogeneity across countries: $\tilde{u}_{it} = \hat{u}_{it}/\hat{\sigma}_{\varepsilon i}$ and $\tilde{v}_{it} = \hat{v}_{it}/\hat{\sigma}_{\varepsilon i}$, where $\hat{\sigma}_{\varepsilon i} = \frac{1}{T-p_i-1} \sum_{t=p_i+2}^{T} (\hat{u}_{it} - \hat{\delta}_i \hat{v}_{it})$, with $\hat{\delta}_i$ obtained by regress $\hat{u}_{it}$ against $\hat{v}_{it-1}$. The second stage of regression is estimating δ

using Ordinary Least Square (OLS) and computing the statistical inference. Specifically,

$$\tilde{u}_{it} = \delta \tilde{v}_{i,t-1} + \tilde{\varepsilon}_{it}.$$

Finally form the adjusted t-statistic and test the significance of δ for unit root. Since this is a brief revisit of Levin et al (2002)'s framework, we will skip the discussion of calculation of inference. At the end, the LLC-2SLS estimated half-life is calculated by $t^* = \ln(.5)/\ln(1+\delta)$.

Appendix B

Supplement to Chapter 4

B.1 Solving dynamic programming problem for representative household

The representative household seeks C_t, M_t/P_t and N_t to maximize its life-time utility

$$E_0 \sum_{t=0}^{\infty} \beta^t H\left(C_t, \frac{M_t}{P_t}, N_t\right).$$

subject to a sequence of flow budget constraints

$$\int_0^1 P_t(i)C_t(i)di + B_t + \int_0^1 M_t(j)dj = Q_{t-1}B_{t-1} + \int_0^1 R_{t-1}(j)M_{t-1}(j)dj + W_tN_t - T_t$$

where $t = 0, 1, 2, ...$

Let $A_t = Q_{t-1}B_{t-1} + \int_0^1 R_{t-1}(j)M_{t-1}(j)dj$ be the state variable, then

$$\begin{aligned} B_t + \int_0^1 M_t(j)dj &= \frac{1}{Q_t}\left[Q_tB_t + \int_0^1 Q_tM_t(j)dj\right] \\ &= \frac{1}{Q_t}\left[Q_tB_t + \int_0^1 R_t(j)M_t(j)dj - \int_0^1 R_t(j)M_t(j)dj + \int_0^1 Q_tM_t(j)dj\right] \\ &= \frac{1}{Q_t}A_{t+1} + \int_0^1 \frac{Q_t - R_t(j)}{Q_t}M_t(j)dj. \end{aligned}$$

Choose control variables as

$$\begin{aligned}
\mu_{1t} &= P_t C_t - A_t + T_t \\
\mu_{2t} &= \int_0^1 \frac{Q_t - R_t(j)}{Q_t} M_t(j) dj, \forall j \in [0,1] \\
\mu_{3t} &= W_t N_t.
\end{aligned}$$

Notice that μ_{2t} is independent of $j \in [0,1]$.

The Law of Motion can be obtained from budget constraint

$$\frac{1}{Q_t} A_{t+1} = -\mu_{1t} - \mu_{2t} + \mu_{3t}$$

Setup Bellman equation for dynamic programming as

$$f_t(A_t) = \max E_t \left\{ H_t \left(C_t, \frac{M_t}{P_t}, N_t \right) + \beta f_{t+1}(A_{t+1}) \right\}.$$

Take derivative of $f_t(A_t)$ yields

$$f_t^{'}(A_t) = \frac{H_{Ct}}{P_t}.$$

Household choose optimal level of μ_{1t}, μ_{2t} and μ_{3t} to yield maximum utility. Therefore one can obtain optimal conditions as

$$\begin{aligned}
0 &= \frac{\partial f_t(A_t)}{\partial \mu_{1t}} = H_{Ct} \frac{\partial C_t}{\partial \mu_{1t}} + H_{Mt} \frac{\partial \left(\frac{M_t}{P_t} \right)}{\partial \mu_{1t}} + H_{Nt} \frac{\partial N_t}{\partial \mu_{1t}} + \beta E_t \left\{ f_t^{'}(A_t) \frac{\partial A_{t+1}}{\partial \mu_{1t}} \right\} \\
0 &= \frac{\partial f_t(A_t)}{\partial \mu_{2t}} = H_{Ct} \frac{\partial C_t}{\partial \mu_{2t}} + H_{Mt} \frac{\partial \left(\frac{M_t}{P_t} \right)}{\partial \mu_{2t}} + H_{Nt} \frac{\partial N_t}{\partial \mu_{2t}} + \beta E_t \left\{ f_t^{'}(A_t) \frac{\partial A_{t+1}}{\partial \mu_{2t}} \right\} \\
0 &= \frac{\partial f_t(A_t)}{\partial \mu_{3t}} = H_{Ct} \frac{\partial C_t}{\partial \mu_{3t}} + H_{Mt} \frac{\partial \left(\frac{M_t}{P_t} \right)}{\partial \mu_{3t}} + H_{Nt} \frac{\partial N_t}{\partial \mu_{3t}} + \beta E_t \left\{ f_t^{'}(A_t) \frac{\partial A_{t+1}}{\partial \mu_{3t}} \right\}
\end{aligned}$$

Using $f_t'(A_t) = H_{Ct}/P_t$, and

$$\begin{aligned}
\frac{\partial C_t}{\partial \mu_{1t}} &= \frac{1}{P_t},\ \frac{\partial C_t}{\partial \mu_{2t}} = 0,\ \frac{\partial C_t}{\partial \mu_{3t}} = 0, \\
\frac{\partial \left(\frac{M_t}{P_t}\right)}{\partial \mu_{1t}} &= 0,\ \frac{\partial \left(\frac{M_t}{P_t}\right)}{\partial \mu_{2t}} = \frac{\partial M_t}{\partial M_t(j)} \frac{Q_t - R_t(j)}{Q_t P_t},\ \frac{\partial \left(\frac{M_t}{P_t}\right)}{\partial \mu_{3t}} = 0, \\
\frac{\partial N_t}{\partial \mu_{1t}} &= 0,\ \frac{\partial N_t}{\partial \mu_{2t}} = 0,\ \frac{\partial N_t}{\partial \mu_{3t}} = \frac{1}{W_t},
\end{aligned}$$

three above optimal conditions will turn into the first order conditions that appear in the main text

$$\begin{aligned}
\frac{H_{Ct}}{P_t} &= Q_t \beta E_t \left\{ \frac{H_{Ct+1}}{P_{t+1}} \right\} \\
\frac{H_{Mt}}{H_{Ct}} &= \frac{Q_t - R_t(j)}{Q_t},\ \forall j \in [0,1] \\
-\frac{H_{Nt}}{H_{Ct}} &= \frac{W_t}{P_t}.
\end{aligned}$$

The first condition is Euler equation which represents household's inter-temporal choice of consumption. The second condition is contemporary money demand function for an arbitrary monetary asset $j \in [0,1]$. The third condition is household's labor supply schedule.

B.2 The duality of user-cost aggregate and monetary aggregate

At any period $t = 1, 2, ...$, household seek $M_t(j)$ for all $j \in [0,1]$ to maximize its monetary aggregate index

$$M_t \equiv \left(\int_0^1 \eta(j)^{\frac{1}{\omega}} M_t(j)^{\frac{\omega-1}{\omega}} dj \right)^{\frac{\omega}{\omega-1}}$$

subject to

$$\int_0^1 U_t(j) M_t(j) dj = Z_t. \tag{B.1}$$

where Z_t is the expenditure of monetary assets and considered as given. $U_t(j) \equiv \frac{Q_t - R_t(j)}{Q_t}$ for all $j \in [0,1]$ is the *user-cost of monetary asset* j. Form Lagrangian and the associated first-order conditions can be obtained as

$$\left(\frac{M_t(j)}{\eta(j)}\right)^{-\frac{1}{\omega}} M_t^{\frac{1}{\omega}} = \lambda_l U_t(j)$$

for all $j \in [0,1]$ where λ_l is the Lagrangian multiplier. Thus, for any two monetary assets (j,k),

$$\frac{M_t(j)}{M_t(k)}\frac{\eta(k)}{\eta(j)} = \left(\frac{U_t(j)}{U_t(k)}\right)^{-\omega}$$

substitute the expression of $M_t(k)$ of the above equation into the monetary assets expenditure budget constraint (B.1) to yield

$$Z_t = \left[\eta(j)^{-1} U_t(j)^{\omega} M_t(j)^{-1}\right] \int_0^1 \eta(k) U_t(k)^{1-\omega} dk.$$

Define *user-cost aggregate index* $U_t \equiv \left[\int_0^1 \eta(k) U_t(k)^{1-\omega} dk\right]^{\frac{1}{1-\omega}}$ and the above equation can be rewritten as

$$\frac{Z_t}{U_t}\left(\frac{U_t}{U_t(j)}\right)^{\omega} = \eta(j)^{-1} M_t(j)$$

for all $j \in [0,1]$. Rearrange the terms

$$\left(\frac{U_t}{Z_t}\right)^{\frac{1}{\omega}}\left(\frac{U_t(j)}{U_t}\right) = \eta(j)^{\frac{1}{\omega}} M_t(j)^{-\frac{1}{\omega}},$$

multiply $M_t(j)$ by both side

$$\left(\frac{U_t}{Z_t}\right)^{\frac{1}{\omega}}\left(\frac{U_t(j) M_t(j)}{U_t}\right) = \eta(j)^{\frac{1}{\omega}} M_t(j)^{1-\frac{1}{\omega}},$$

and summing through $j \in [0,1]$, using the definition of monetary aggregate $M_t \equiv \left(\int_0^1 \eta(j)^{\frac{1}{\omega}} M_t(j)^{\frac{\omega-1}{\omega}} dj \right)^{\frac{\omega}{\omega-1}}$, we can finally obtain

$$\int_0^1 U_t(j) M_t(j) dj = Z_t = U_t M_t.$$

In other words, user-cost aggregate U_t and monetary aggregate M_t are dual.

B.3 Aggregate money demand function (proof of Proposition 4)

From definition of marginal utility of individual monetary asset $H_{M,t}(j)$ we have

$$H_{M,t}(j) \frac{M_t(j)}{M_t} = H_{M,t} \frac{\partial M_t}{\partial M_t(j)} \frac{M_t(j)}{M_t}, \ \forall j \in [0,1].$$

Summing through $j \in [0,1]$ we will get

$$\int_0^1 H_{M,t}(j) \frac{M_t(j)}{M_t} dj = H_{M,t} \int_0^1 \frac{\partial M_t}{\partial M_t(j)} \frac{M_t(j)}{M_t} dj. \tag{B.2}$$

Notice that $\frac{\partial M_t}{\partial M_t(j)} \frac{M_t(j)}{M_t}$ is the elasticity of monetary aggregate M_t with respect to an arbitrary individual monetary asset $M_t(j)$. From the functional form of theoretical monetary aggregate (4.6) we can obtain

$$\begin{aligned} 1 &\equiv \int_0^1 \eta(j)^{\frac{1}{\omega}} \left(\frac{M_t(j)}{M_t} \right)^{\frac{\omega-1}{\omega}} dj \\ &= \int_0^1 \eta(j)^{\frac{1}{\omega}} \left(\frac{M_t}{M_t(j)} \right)^{\frac{1}{\omega}} \frac{M_t(j)}{M_t} dj \ . \end{aligned}$$

Since

$$\frac{\partial M_t}{\partial M_t(j)} = \left[\eta(j) \frac{M_t}{M_t(j)} \right]^{\frac{1}{\omega}}$$

we shall get

$$1 = \int_0^1 \frac{\partial M_t}{\partial M_t(j)} \frac{M_t(j)}{M_t} dj.$$

In other words, elasticity of monetary aggregate M_t with respect to an arbitrary individual monetary asset $M_t(j)$ can be treated as a weight.

Now (B.2) turns to

$$\int_0^1 H_{M,t}(j) \frac{M_t(j)}{M_t} dj = H_{M,t}.$$

Substitute household's optimal condition (4.2) into the above equation and rearrange the terms,

$$\frac{\int_0^1 U_t(j) M_t(j) dj}{M_t} = \frac{H_{Mt}}{H_{Ct}}.$$

We have shown that $\int_0^1 U_t(j) M_t(j) dj = U_t M_t$. Therefore, the above equation turns to

$$U_t = \frac{H_{Mt}}{H_{Ct}}$$

which can be viewed as an aggregate level money demand function: the household optimal decision for balance holding of monetary asset is settled at the point where the ratio of marginal utility of monetary aggregate and marginal utility of consumption is equal to U_t. In our context, the user-cost of individual monetary asset, $U_t(j)$, is viewed as the "price" of holding monetary asset j. Therefore, the user-cost of aggregate monetary asset, U_t, can be viewed as the "price" of holding monetary aggregate. Using the utility specification, one can finally arrive at

$$U_t = \left(\frac{\theta}{1-\theta} \right) \left(\frac{P_t C_t}{M_t} \right)^{\nu}$$

which is (4.9).

B.4 Log-linearization of first order conditions

We have obtained three optimal conditions (4.4), (4.9) and (4.5). We copy them below for reader's convenience,

$$\begin{aligned}
\frac{1}{Q_t} &= \beta E_t \left\{ \left(\frac{X_{t+1}}{X_t} \right)^{\nu-\sigma} \left(\frac{C_{t+1}}{C_t} \right)^{-\nu} \frac{P_t}{P_{t+1}} \right\} \\
U_t &= \left(\frac{\theta}{1-\theta} \right) \left(\frac{P_t C_t}{M_t} \right)^{\nu} \\
\frac{W_t}{P_t} &= \frac{N_t^{\varphi}}{(1-\theta) X_t^{\nu-\sigma} C_t^{-\nu}}.
\end{aligned}$$

Before starting log-linearization, we first illustrate one useful identity. Let lower case letter stands for the logarithm of the associate upper case letter. Specifically, $z_t \equiv \log Z_t$. Define a (log) variable's deviation from its (log) steady state as

$$\widetilde{z}_t \equiv \log Z_t - \log Z$$

where Z is the steady state of variable Z_t. Then it is obvious that

$$Z_t \equiv Z e^{\widetilde{z}_t}.$$

Since a variable's deviation from its steady state is small, first-order Taylor expansion of the right hand side around steady state Z yields

$$Z_t \approx Z \left(1 + \widetilde{z}_t \right).$$

Using the identity above, log-linearize optimal conditions (4.4), (4.9) and (4.5) to yield

$$\begin{aligned}
0 &= \widetilde{i}_t - \sigma E_t \left\{\triangle\widetilde{c}_{t+1}\right\} - (\nu - \sigma) E_t \left\{\triangle\widetilde{c}_{t+1} - \triangle\widetilde{x}_{t+1}\right\} - E_t \left\{\widetilde{\pi}_{t+1}\right\} \\
\widetilde{m}_t - \widetilde{p}_t &= \widetilde{c}_t - \frac{1}{\nu}\widetilde{u}_t \\
\widetilde{w}_t - \widetilde{p}_t &= \sigma\widetilde{c}_t + \varphi\widetilde{n}_t + (\nu - \sigma)\left(\widetilde{c}_t - \widetilde{x}_t\right).
\end{aligned}$$

Next we will show that $(\widetilde{c}_t - \widetilde{x}_t)$ is a function of $\widetilde{c}_t$, $\widetilde{m}_t$ and $\widetilde{p}_t$. Start with the consumption-real balance aggregate

$$X_t \equiv \left[(1-\theta)C_t^{1-\nu} + \theta\left(\frac{M_t}{P_t}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}}$$

We shall then have

$$X_t^{1-\nu} \equiv (1-\theta)C_t^{1-\nu} + \theta\left(\frac{M_t}{P_t}\right)^{1-\nu} \tag{B.3}$$

and its steady states satisfy

$$X^{1-\nu} \equiv (1-\theta)C^{1-\nu} + \theta\left(\frac{M}{P}\right)^{1-\nu}.$$

Log-linearize above equation around its steady states yields

$$X^{1-\nu}\widetilde{x}_t = (1-\theta)C^{1-\nu}\widetilde{c}_t + \theta\left(\frac{M}{P}\right)^{1-\nu}\left(\widetilde{m}_t - \widetilde{p}_t\right).$$

Rearrange the terms

$$\widetilde{x}_t = \frac{(1-\theta)C^{1-\nu}}{X^{1-\nu}}\widetilde{c}_t + \frac{\theta\left(\frac{M}{P}\right)^{1-\nu}}{X^{1-\nu}}\left(\widetilde{m}_t - \widetilde{p}_t\right).$$

Since at the steady state

$$\frac{(1-\theta)C^{1-\nu}}{X^{1-\nu}} = 1 - \frac{\theta\left(\frac{M}{P}\right)^{1-\nu}}{X^{1-\nu}},$$

we shall have

$$
\begin{aligned}
\widetilde{c}_t - \widetilde{x}_t &= \frac{\theta\left(\frac{M}{P}\right)^{1-\nu}}{X^{1-\nu}}\widetilde{c}_t - \frac{\theta\left(\frac{M}{P}\right)^{1-\nu}}{X^{1-\nu}}\left(\widetilde{m}_t - \widetilde{p}_t\right) \\
&= \frac{\theta\left(\frac{M}{P}\right)^{1-\nu}}{X^{1-\nu}}\left[\widetilde{c}_t - \left(\widetilde{m}_t - \widetilde{p}_t\right)\right].
\end{aligned}
$$

Notice the inverse of the coefficient of the right hand side of the second equation can be expressed as

$$
\begin{aligned}
\frac{X^{1-\nu}}{\theta\left(\frac{M}{P}\right)^{1-\nu}} &= 1 + \frac{(1-\theta)C^{1-\nu}}{\theta\left(\frac{M}{P}\right)^{1-\nu}} \\
&= 1 + \frac{1-\theta}{\theta}\left(\frac{C}{\frac{M}{P}}\right)^{1-\nu}
\end{aligned}
$$

From aggregate money demand (4.9), we can have its steady state

$$
U = \left(\frac{\theta}{1-\theta}\right)\left(\frac{PC}{M}\right)^{\nu}
$$

then

$$
U^{1-\nu} = \left(\frac{\theta}{1-\theta}\right)^{1-\nu}\left(\frac{PC}{M}\right)^{\nu(1-\nu)}
$$

rearrange terms

$$
\frac{1-\theta}{\theta}\left(\frac{PC}{M}\right)^{1-\nu} = \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{\nu}} U^{\frac{1-\nu}{\nu}}.
$$

Therefore,

$$
\frac{\theta\left(\frac{M}{P}\right)^{1-\nu}}{X^{1-\nu}} = \frac{1}{1 + \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{\nu}} U^{\frac{1-\nu}{\nu}}}
$$

We define

$$
\chi \equiv \frac{1}{1 + \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{\nu}} U^{\frac{1-\nu}{\nu}}}
$$

and let $V = \frac{C}{M/P}$ be the velocity, then $\chi \equiv \frac{U}{V+U}$, and $0 < \chi < 1$.

Further,

$$\begin{aligned} \widetilde{c}_t - \widetilde{x}_t &= \chi \left[\widetilde{c}_t - (\widetilde{m}_t - \widetilde{p}_t)\right] \\ &= \frac{\chi}{\nu} \widetilde{u}_t \end{aligned}$$

and the second equality used the money demand $\widetilde{m}_t - \widetilde{p}_t = \widetilde{c}_t - \frac{1}{\nu}\widetilde{u}_t$. Therefore the first order conditions can be rewritten as

$$\begin{aligned} 0 &= \widetilde{i}_t - E_t\left\{\widetilde{\pi}_{t+1}\right\} - \sigma E_t\left\{\triangle \widetilde{c}_{t+1}\right\} - \frac{\chi}{\nu}\left(\nu - \sigma\right) E_t\left\{\triangle \widetilde{u}_{t+1}\right\} \\ \widetilde{m}_t - \widetilde{p}_t &= \widetilde{c}_t - \frac{1}{\nu}\widetilde{u}_t \\ \widetilde{w}_t - \widetilde{p}_t &= \sigma \widetilde{c}_t + \varphi \widetilde{n}_t + \frac{\chi}{\nu}\left(\nu - \sigma\right)\widetilde{u}_t. \end{aligned}$$

In terms of log variables instead of their deviation from steady states,

$$c_t - x_t = \chi\left[c_t - (m_t - p_t)\right] + \Sigma$$

where $\Sigma \equiv \frac{1}{1-\nu}\log\left(\frac{1-\chi}{1-\theta}\right) - \frac{\chi}{\nu}\log\left(\frac{1-\theta}{\theta}U\right)$. Then the optimal conditions can be written as

$$\begin{aligned} i_t - E_t\{\pi_{t+1}\} - \rho &= \sigma E_t\left\{\Delta c_{t+1}\right\} + \frac{\chi}{\nu}\left(\nu - \sigma\right) E_t\left\{\Delta u_{t+1}\right\} \\ m_t - p_t &= c_t - \frac{1}{\nu}u_t + \frac{1}{\nu}\log\left(\frac{\theta}{1-\theta}\right) \\ w_t - p_t &= \sigma c_t + \varphi n_t + \frac{\chi}{\nu}\left(\nu - \sigma\right) u_t + \frac{\chi}{\nu}\left(\nu - \sigma\right)\log\left(\frac{\theta}{1-\theta}\right) \\ &\quad + \left(\nu - \sigma\right)\Sigma - \log\left(1 - \theta\right). \end{aligned}$$

We eliminate non-interested constant terms (the steady states),

$$\begin{aligned} i_t - E_t\{\pi_{t+1}\} - \rho &= \sigma E_t\{\Delta c_{t+1}\} - \frac{\chi}{\nu}(\nu - \sigma) E_t\{\Delta u_{t+1}\} \\ m_t - p_t &= c_t - \frac{1}{\nu} u_t \\ w_t - p_t &= \sigma c_t + \varphi n_t + \frac{\chi}{\nu}(\nu - \sigma) u_t \end{aligned}$$

B.5 Moneyness non-neutrality on natural output

We have claimed that under this paper's model setup, natural output y_t^n depends on technology and real balance of monetary aggregate. Specifically,

$$\Delta y_t^n = \tau_a \Delta a_t + \tau_m \Delta m_t$$

where

$$\begin{aligned} \tau_a &\equiv \frac{\frac{1+\varphi}{1-\alpha}}{(\sigma + \lambda) + \frac{\varphi+\alpha}{1-\alpha}} > 0 \\ \tau_m &\equiv \frac{\lambda}{\sigma + \lambda + \frac{\varphi+\alpha}{1-\alpha}}. \end{aligned}$$

We will now formally show this property. The economy's average real marginal cost can be expressed as

$$mc_t = (w_t - p_t) - mpn_t \tag{B.4}$$

where mpn_t stands for economy's (log) marginal product of labor. Using economy's production function $Y_t = A_t {N_t}^{1-\alpha}$ we can obtain that

$$\begin{aligned} mpn_t &= a_t - \alpha n_t + \log(1-\alpha) \\ &= y_t - n_t + \log(1-\alpha). \end{aligned}$$

Using above expression of mpn_t and household's labor supply schedule

$$w_t - p_t = \sigma c_t + \varphi n_t + \frac{\chi}{\nu}(\nu - \sigma) u_t + \frac{\chi}{\nu}(\nu - \sigma)\log\left(\frac{\theta}{1-\theta}\right) + (\nu - \sigma)\Sigma - \log(1-\theta),$$

we can rewrite the average marginal cost (B.4) as

$$\begin{aligned} mc_t &= (w_t - p_t) - mpn_t \\ &= \left[\sigma c_t + \varphi n_t + \frac{\chi}{\nu}(\nu - \sigma) u_t + \frac{\chi}{\nu}(\nu - \sigma)\log\left(\frac{\theta}{1-\theta}\right) + (\nu - \sigma)\Sigma - \log(1-\theta)\right] \\ &\quad - [y_t - n_t + \log(1-\alpha)] \\ &= \sigma c_t + \varphi n_t + \frac{\chi}{\nu}(\nu - \sigma) u_t - (y_t - n_t) + \\ &\quad \left[\frac{\chi}{\nu}(\nu - \sigma)\log\left(\frac{\theta}{1-\theta}\right) + (\nu - \sigma)\Sigma - \log(1-\theta) - \log(1-\alpha)\right] \end{aligned}$$

Define $\lambda \equiv \chi(\nu - \sigma)$ where $\chi \equiv \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{\nu}} U^{\frac{1-\nu}{\nu}}\right]^{-1}$ and using commodity market clear condition, one can obtain

$$mc_t = (\sigma - 1) y_t + (\varphi + 1) n_t + \frac{\lambda}{\nu} u_t + F$$

where

$$F \equiv \frac{\chi}{\nu}(\nu - \sigma)\log\left(\frac{\theta}{1-\theta}\right) + (\nu - \sigma)\Sigma - \log(1-\theta) - \log(1-\alpha).$$

Using household's money demand $m_t - p_t = c_t - \frac{1}{\nu} u_t$ and rearrange terms

$$mc_t = (\lambda + \sigma - 1) y_t + (\varphi + 1) n_t - \lambda(m_t - p_t) + F.$$

Lemma 7 *Under the assumption of zero steady state inflation rate, household's labor supply n_t is a function of output y_t and technology level a_t up to the second order. Specifically*

$$n_t = \frac{1}{1-\alpha}(y_t - a_t)$$

Proof. *See Gali (2008) page 46 and page 62-63.* ■

Now the economy's average real marginal cost can be written as

$$
\begin{aligned}
mc_t &= (\lambda + \sigma - 1)y_t + \frac{\varphi + 1}{1 - \alpha}(y_t - a_t) - \lambda(m_t - p_t) + \digamma \\
&= (\lambda + \sigma + \frac{\varphi + \alpha}{1 - \alpha})y_t - \frac{1 + \varphi}{1 - \alpha}a_t - \lambda(m_t - p_t) + \digamma
\end{aligned}
$$

It can be shown that under flexible price-settings, economy's marginal cost is time-invariant $mc = \log \frac{\varepsilon - 1}{\varepsilon}$. Define the *natural output* y_t^n as the equilibrium level of output under flexible price, then it turns that

$$
mc = (\lambda + \sigma + \frac{\varphi + \alpha}{1 - \alpha})y_t^n - \frac{1 + \varphi}{1 - \alpha}a_t - \lambda m_t + (\digamma + \lambda p^*)
$$

where p^* is the steady state price level. Take difference of the above equation and rearrange terms

$$
\Delta y_t^n = \tau_a \Delta a_t + \tau_m \Delta m_t
$$

where

$$
\begin{aligned}
\tau_a &\equiv \frac{\frac{1+\varphi}{1-\alpha}}{(\sigma + \lambda) + \frac{\varphi+\alpha}{1-\alpha}} > 0 \\
\tau_m &\equiv \frac{\lambda}{(\sigma + \lambda) + \frac{\varphi+\alpha}{1-\alpha}}.
\end{aligned}
$$

Coefficient τ_a is positive. However, the sign of τ_m is ambiguous. Since denominator of τ_m is positive, the sign of τ_m is determined by $\lambda \equiv \chi(\nu - \sigma)$ which in turn by $(\nu - \sigma)$. Then if $\nu > \sigma$, or elasticity of substitution for intra-temporal is greater than for inter-temporal, $\tau_m > 0$ and real balance has a positive effect on natural output. But if $\nu < \sigma$, or elasticity of substitution for intra-temporal is smaller than for inter-temporal, $\tau_m < 0$ and real balance has a negative effect on natural output. A special case is that when $\nu = \sigma$, we will have $\tau_m = 0$ and potential output is immune from real balance.

B.6 The Dynamic System of Economy

We have obtained the New-Keynesian Pillips Curve (NKPC):

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \kappa \widehat{y}_t$$

where $\kappa \equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}(1-\beta\theta)\left(\frac{1-\theta}{\theta}\right)\left(\sigma+\lambda+\frac{\alpha+\varphi}{1-\alpha}\right) > 0$, and the dynamic IS (DIS) curve:

$$\widehat{y}_t = E_t\{\widehat{y}_{t+1}\} - \frac{1}{\sigma+\lambda}\left[i_t - \rho - (1+\lambda)E_t\{\pi_{t+1}\} + \lambda E_t\{\Delta m_{t+1}\}\right] + E_t\{\Delta y^n_{t+1}\}.$$

Now we are trying to derive the compact matrix form of the dynamic system for the economy using the two above equations. From the DIS curve we can have

$$0 = (\sigma+\lambda)\,E_t\{\Delta y_{t+1}\} - \widetilde{i}_t + (1+\lambda)\,E_t\{\pi_{t+1}\} - \lambda E_t\{\Delta m_{t+1}\}$$

Insert bond rate determine equation (4.18) into the above,

$$\begin{aligned} 0 &= (\sigma+\lambda)\,E_t\{\Delta y_{t+1}\} - \lambda E_t\{\Delta m_{t+1}\} + (1+\lambda)\,E_t\{\pi_{t+1}\} \\ &\quad -\phi_y E_t\{\Delta y_{t+1}\} + \phi_m E_t\{\Delta m_{t+1}\} - \phi_\pi E_t\{\Delta \pi_{t+1}\} - \varepsilon_{i,t} \\ &= (\sigma+\lambda-\phi_y)\,E_t\{\Delta y_{t+1}\} - (\lambda-\phi_m)\,E_t\{\Delta m_{t+1}\} + (1+\lambda-\phi_\pi)\,E_t\{\pi_{t+1}\} - \varepsilon_{i,t} \end{aligned}$$

rearrange the terms,

$$\widehat{y}_t = E_t\{\widehat{y}_{t+1}\} - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}E_t\{\Delta m_{t+1}\} + \frac{1+\lambda-\phi_\pi}{\sigma+\lambda-\phi_y}E_t\{\pi_{t+1}\} - \frac{1}{\sigma+\lambda-\phi_y}\varepsilon_{i,t} + E_t\{\Delta y^n_{t+1}\}.$$

Combining the above equation with the natural output determination (4.15),

$$\begin{aligned}
\widehat{y}_t &= E_t\{\widehat{y}_{t+1}\} - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}E_t\{\Delta m_{t+1}\} + \frac{1+\lambda-\phi_\pi}{\sigma+\lambda-\phi_y}E_t\{\pi_{t+1}\} - \frac{1}{\sigma+\lambda-\phi_y}\varepsilon_{i,t} \\
&\quad + E_t\{\tau_a \Delta a_{t+1}\} + E_t\{\tau_m \Delta m_{t+1}\} \\
&= E_t\{\widehat{y}_{t+1}\} + \left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)E_t\{\Delta m_{t+1}\} + \frac{1+\lambda-\phi_\pi}{\sigma+\lambda-\phi_y}E_t\{\pi_{t+1}\} \\
&\quad + \tau_a(\phi_a - 1)a_t - \frac{1}{\sigma+\lambda-\phi_y}\varepsilon_{i,t}
\end{aligned}$$

further use the central bank monetary policy rule (4.21) to substitute $E_t\{\Delta m_{t+1}\}$ we have

$$\begin{aligned}
\widehat{y}_t &= E_t\{\widehat{y}_{t+1}\} + \frac{1+\lambda-\phi_\pi}{\sigma+\lambda-\phi_y}E_t\{\pi_{t+1}\} + \tau_a(\phi_a-1)a_t - \frac{1}{\sigma+\lambda-\phi_y}\varepsilon_{i,t} \\
&\quad + \left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)E_t\{\varphi_{y0}\widehat{y}_{t+1} + \varphi_{y1}\widehat{y}_t + \varphi_\pi \pi_{t+1} + \zeta_{t+1}\}
\end{aligned}$$

rearrange the terms,

$$\begin{aligned}
\left[1 - \left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\varphi_{y_1}\right]\widehat{y}_t &= \left[1 + \left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\varphi_{y_0}\right]E_t\{\widehat{y}_{t+1}\} \\
&\quad + \left[\frac{1+\lambda-\phi_\pi}{\sigma+\lambda-\phi_y} + \left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\varphi_\pi\right]E_t\{\pi_{t+1}\} \\
&\quad + \left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\phi_\zeta \zeta_t + \tau_a(\phi_a-1)a_t - \frac{1}{\sigma+\lambda-\phi_y}\varepsilon_{i,t}
\end{aligned}$$

For the convenience of analysis, we define a few more parameters. Let

$$\begin{aligned}
A &= 1 - \left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\varphi_{y_1} \\
B &= 1 + \left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\varphi_{y_0} \\
C &= \frac{1+\lambda-\phi_\pi}{\sigma+\lambda-\phi_y} + \left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\varphi_\pi \\
S_t &= \left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\phi_\zeta \zeta_t + \tau_a(\phi_a-1)a_t - \frac{1}{\sigma+\lambda-\phi_y}\varepsilon_{i,t}
\end{aligned}$$

then we have

$$A\widehat{y}_t = BE_t\{\widehat{y}_{t+1}\} + CE_t\{\pi_{t+1}\} + S_t.$$

Substitute the above equation into NKPC,

$$\begin{aligned}
\pi_t &= \beta E_t\{\pi_{t+1}\} + \kappa\left[\frac{B}{A}E_t\{\widehat{y}_{t+1}\} + \frac{C}{A}E_t\{\pi_{t+1}\} + \frac{S_t}{A}\right] \\
&= \kappa\frac{B}{A}E_t\{\widehat{y}_{t+1}\} + \left(\beta + \kappa\frac{C}{A}\right)E_t\{\pi_{t+1}\} + \frac{\kappa}{A}S_t
\end{aligned}$$

Finally write the system into matrix form,

$$\begin{pmatrix} \widehat{y}_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} \frac{B}{A} & \frac{C}{A} \\ \kappa\frac{B}{A} & \beta + \kappa\frac{C}{A} \end{pmatrix} \begin{pmatrix} E_t\{\widehat{y}_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{pmatrix} + S_t \begin{pmatrix} \frac{1}{A} \\ \frac{\kappa}{A} \end{pmatrix}.$$

The charactoristic polynomial is given by $\xi I - \begin{pmatrix} \frac{B}{A} & \frac{C}{A} \\ \kappa\frac{B}{A} & \beta + \kappa\frac{C}{A} \end{pmatrix}$. Take determinate of the charactoristic polynomial we have

$$\xi^2 - \left(\frac{B}{A} + \beta + \kappa\frac{C}{A}\right)\xi + \beta\frac{B}{A}.$$

The dynamic system have local unique solution is equivalent to that the coefficient matrix has both of its eigenvalues within the unit circle. LaSelle (1986) showed that the following necessary and sufficient conditions must be satisfied,

$$\begin{aligned}
\left|\beta\frac{B}{A}\right| &< 1 \\
\left|\frac{B}{A} + \beta + \kappa\frac{C}{A}\right| &< 1 + \beta\frac{B}{A}.
\end{aligned}$$

B.7 The Impulse Response Functions

B.7.1 The Impulse Response For Monetary Policy Shock

The economy's forward-looking dynamic system of NKPC (4.16), DIS (4.17), and policy rule (4.21) can be written more compactly as a system of two-variable, simultaneous equations

$$\begin{pmatrix} \widehat{y}_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} \frac{B}{A} & \frac{C}{A} \\ \kappa\frac{B}{A} & \beta + \kappa\frac{C}{A} \end{pmatrix} \begin{pmatrix} E_t\left\{\widehat{y}_{t+1}\right\} \\ E_t\left\{\pi_{t+1}\right\} \end{pmatrix} + S_t \begin{pmatrix} \frac{1}{A} \\ \frac{\kappa}{A} \end{pmatrix}$$

where

$$\begin{aligned} A &= 1 - \left(\tau_m - \frac{\lambda - \phi_m}{\sigma + \lambda - \phi_y}\right)\varphi_{y_1} \\ B &= 1 + \left(\tau_m - \frac{\lambda - \phi_m}{\sigma + \lambda - \phi_y}\right)\varphi_{y_0} \\ C &= \frac{1 + \lambda - \phi_\pi}{\sigma + \lambda - \phi_y} + \left(\tau_m - \frac{\lambda - \phi_m}{\sigma + \lambda - \phi_y}\right)\varphi_\pi \\ S_t &= \left(\tau_m - \frac{\lambda - \phi_m}{\sigma + \lambda - \phi_y}\right)\phi_\zeta\zeta_t + \tau_a(\phi_a - 1)a_t - \frac{1}{\sigma + \lambda - \phi_y}\varepsilon_{i,t} \end{aligned}$$

The components of S_t shows that the shock terms contains three components, exogenous technology a_t, exogenously determined benchmark asset shock $\varepsilon_{i,t}$, and monetary policy shock ζ_t which is determined by the central bank.

To derive the impulse response function with respect to monetary policy shock, we first turn off the financial institutional shock and technology shock so that $\varepsilon_{i,t} = 0$ and $a_t = 0$. Then we shall have a difference equation system

$$\begin{aligned} \widehat{y}_t &= \frac{B}{A}E_t\left\{\widehat{y}_{t+1}\right\} + \frac{C}{A}E_t\left\{\pi_{t+1}\right\} + \frac{1}{A}S_t \\ \pi_t &= \beta E_t\left\{\pi_{t+1}\right\} + \kappa\widehat{y}_t \end{aligned}$$

where $S_t = \left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\phi_\zeta\zeta_t$.

Utilizing the undetermined coefficient method and guess the solutions of the system take the form

$$\begin{aligned}\widehat{y}_t &= M\zeta_t \\ \pi_t &= N\zeta_t\end{aligned}$$

where M and N are the coefficients to be determined. Since we have assumed that monetary policy shock ζ_t is a first order auto-regressive stochastic process,

$$\zeta_t = \phi_\zeta \zeta_{t-1} + \varepsilon_{\zeta,t}$$

for $t = 1, 2, 3, ...$ and $\phi_\zeta \in [0, 1)$. $\varepsilon_{\zeta,t}$ is assumed to be $i.i.d.$ with mean 0. Take the above guessed solution system one period forward and take expectations conditioning on period t, we then have

$$\begin{aligned}E_t\left\{\widehat{y}_{t+1}\right\} &= M\phi_\zeta\zeta_t \\ E_t\left\{\pi_{t+1}\right\} &= N\phi_\zeta\zeta_t\end{aligned}$$

Then, we can solve the system for coefficient M and N by substitute the guessed solutions and their nexted period conditional expectations into the original difference equations system, resulting

$$\begin{aligned}M &= \frac{\left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\phi_\zeta}{A - B\phi_\zeta - \frac{\kappa C\phi_\zeta}{1-\beta\phi_\zeta}} \\ N &= \frac{\left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\phi_\zeta}{A - B\phi_\zeta - \frac{\kappa C\phi_\zeta}{1-\beta\phi_\zeta}}\frac{\kappa}{1-\beta\phi_\zeta}\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial \widehat{y}_t}{\partial \zeta_t} &= \frac{\left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\phi_\zeta}{A - B\phi_\zeta - \frac{\kappa C\phi_\zeta}{1-\beta\phi_\zeta}} \\
\frac{\partial \pi_t}{\partial \zeta_t} &= \frac{\left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\phi_\zeta}{A - B\phi_\zeta - \frac{\kappa C\phi_\zeta}{1-\beta\phi_\zeta}} \frac{\kappa}{1-\beta\phi_\zeta}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{\partial \widehat{y}_{t+s}}{\partial \zeta_t} &= \frac{\partial \widehat{y}_{t+s}}{\partial \zeta_{t+s}} \cdot \frac{\partial \zeta_{t+s}}{\partial \zeta_t} = \frac{\left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\phi_\zeta}{A - B\phi_\zeta - \frac{\kappa C\phi_\zeta}{1-\beta\phi_\zeta}} \cdot \phi_\zeta^s \\
\frac{\partial \pi_{t+s}}{\partial \zeta_t} &= \frac{\partial \pi_{t+s}}{\partial \zeta_{t+s}} \cdot \frac{\partial \zeta_{t+s}}{\partial \zeta_t} = \frac{\left(\tau_m - \frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\phi_\zeta}{A - B\phi_\zeta - \frac{\kappa C\phi_\zeta}{1-\beta\phi_\zeta}} \frac{\kappa}{1-\beta\phi_\zeta} \cdot \phi_\zeta^s
\end{aligned}$$

for $s = 1, 2, 3, ...$

B.7.2 Impulse Response of Output with respect to Monetary Policy Shock

By using the definition of output gap $\widehat{y}_t \equiv y_t - y_t^n$, and by turning off the technology shock, we can obtain

$$\begin{aligned}
\Delta y_t &= \Delta y_t^n + \Delta \widehat{y}_t \\
&= \tau_m \Delta m_t + \Delta \widehat{y}_t \\
&= \tau_m \left(\varphi_{y_0}\widehat{y}_t + \varphi_{y_1}\widehat{y}_{t-1} + \varphi_\pi \pi_t + \zeta_t\right) + \left(\widehat{y}_t - \widehat{y}_{t-1}\right) \\
&= \left(\tau_m \varphi_{y_0} + 1\right)\widehat{y}_t + \left(\tau_m \varphi_{y_1} - 1\right)\widehat{y}_{t-1} + \tau_m \varphi_\pi \pi_t + \tau_m \zeta_t
\end{aligned}$$

Recursively solve for y_t we can have

$$\begin{aligned} y_t &= (\tau_m\varphi_{y_0}+1)\sum_{s=0}^{t-1}\widehat{y}_{t-s}+(\tau_m\varphi_{y_1}-1)\sum_{s=0}^{t-1}\widehat{y}_{t-s-1} \\ &+\tau_m\varphi_\pi\sum_{s=0}^{t-1}\pi_{t-s}+\tau_m\sum_{s=1}^{t}\zeta_s. \end{aligned}$$

Therefore, taking derivative of y_t with respect to ζ_t

$$\frac{\partial y_t}{\partial\zeta_t}=(\tau_m\varphi_{y_0}+1)\frac{\partial\widehat{y}_t}{\partial\zeta_t}+\tau_m\varphi_\pi\frac{\partial\pi_t}{\partial\zeta_t}+\tau_m$$

Furthermore,

$$\begin{aligned} \frac{\partial y_{t+s}}{\partial\zeta_t} &= (\tau_m\varphi_{y_0}+1)\sum_{q=0}^{s}\frac{\partial\widehat{y}_{t+s-q}}{\partial\zeta_t}+(\tau_m\varphi_{y_1}-1)\sum_{q=0}^{s}\frac{\partial\widehat{y}_{t+s-q-1}}{\partial\zeta_t} \\ &+\tau_m\varphi_\pi\sum_{q=0}^{s}\frac{\partial\pi_{t+s-q}}{\partial\zeta_t}+\tau_m\sum_{q=0}^{s}\phi_\zeta^q \end{aligned}$$

for $s=1,2,3,...$, where $\tau_m\equiv\frac{\lambda}{(\sigma+\lambda)+\frac{\varphi+\alpha}{1-\alpha}}$.

B.7.3 Impulse Response of Natural Output with respect to Monetary Policy Shock

We can derive the impulse response of natural output by setting $a_t=0$ we have $\Delta y_t^n=\tau_m\Delta m_t$, then

$$\begin{aligned} \frac{\partial y_t^n}{\partial\zeta_t} &= \frac{\partial y_t}{\partial\zeta_t}-\frac{\partial\widehat{y}_t}{\partial\zeta_t} \\ &= \tau_m\varphi_{y_0}\frac{\partial\widehat{y}_t}{\partial\zeta_t}+\tau_m\varphi_\pi\frac{\partial\pi_t}{\partial\zeta_t}+\tau_m \end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{\partial y_{t+s}^{n}}{\partial \zeta_t} &= \frac{\partial y_{t+s}}{\partial \zeta_t} - \frac{\partial \widehat{y}_{t+s}}{\partial \zeta_t} \\
&= (\tau_m \varphi_{y_0} + 1) \sum_{q=0}^{s} \frac{\partial \widehat{y}_{t+s-q}}{\partial \zeta_t} + (\tau_m \varphi_{y_1} - 1) \sum_{q=0}^{s} \frac{\partial \widehat{y}_{t+s-q-1}}{\partial \zeta_t} \\
&\quad + \tau_m \varphi_\pi \sum_{q=0}^{s} \frac{\partial \pi_{t+s-q}}{\partial \zeta_t} + \tau_m \sum_{q=0}^{s} \phi_\zeta^q - \frac{\partial \widehat{y}_{t+s}}{\partial \zeta_t}
\end{aligned}$$

for $s = 1, 2, 3, ...$, where $\tau_m \equiv \frac{\lambda}{(\sigma+\lambda)+\frac{\varphi+\alpha}{1-\alpha}}$.

B.7.4 Impulse Response of Real Output with respect to Monetary Policy Shock

We shall have

$$\begin{aligned}
\Delta y_t - \pi_t &= \left[(\tau_m \varphi_{y_0} + 1) \widehat{y}_t + (\tau_m \varphi_{y_1} - 1) \widehat{y}_{t-1} + \tau_m \varphi_\pi \pi_t + \tau_m \zeta_t \right] - \pi_t \\
&= (\tau_m \varphi_{y_0} + 1) \widehat{y}_t + (\tau_m \varphi_{y_1} - 1) \widehat{y}_{t-1} + (\tau_m \varphi_\pi - 1) \tau_m \varphi_\pi \pi_t + \tau_m \zeta_t
\end{aligned}$$

Recursivly solve for $y_t - p_t$ we can have

$$\begin{aligned}
y_t - p_t &= (\tau_m \varphi_{y_0} + 1) \sum_{s=0}^{t-1} \widehat{y}_{t-s} + (\tau_m \varphi_{y_1} - 1) \sum_{s=0}^{t-1} \widehat{y}_{t-s-1} \\
&\quad + (\tau_m \varphi_\pi - 1) \sum_{s=0}^{t-1} \pi_{t-s} + \tau_m \sum_{s=1}^{t} \zeta_s .
\end{aligned}$$

Therefore

$$\frac{\partial (y_t - p_t)}{\partial \zeta_t} = (\tau_m \varphi_{y_0} + 1) \frac{\partial \widehat{y}_t}{\partial \zeta_t} + (\tau_m \varphi_\pi - 1) \frac{\partial \pi_t}{\partial \zeta_t} + \tau_m$$

Furthermore,

$$\begin{aligned}
\frac{\partial\left(y_{t+s}-p_{t+s}\right)}{\partial\zeta_t} &= \left(\tau_m\varphi_{y_0}+1\right)\sum_{q=0}^{s}\frac{\partial\widehat{y}_{t+s-q}}{\partial\zeta_t}+\left(\tau_m\varphi_{y_1}-1\right)\sum_{q=0}^{s}\frac{\partial\widehat{y}_{t+s-q-1}}{\partial\zeta_t} \\
&\quad +\left(\tau_m\varphi_\pi-1\right)\sum_{q=0}^{s}\frac{\partial\pi_{t+s-q}}{\partial\zeta_t}+\tau_m\sum_{q=0}^{s}\phi_\zeta^q
\end{aligned}$$

for $s=0,1,2,3,...$, where $\tau_m\equiv\frac{\lambda}{(\sigma+\lambda)+\frac{\varphi+\alpha}{1-\alpha}}$.

B.7.5 Impulse Response of Employment with respect to Monetary Policy Shock

We have shown that $n_t=\frac{1}{1-\alpha}\left(y_t-a_t\right)$. By setting $a_t=0$ we can obtain

$$\begin{aligned}
\frac{\partial n_t}{\partial\zeta_t} &= \frac{\partial n_t}{\partial y_t}\frac{\partial y_t}{\partial\zeta_t} \\
&= \frac{1}{1-\alpha}\frac{\partial y_t}{\partial\zeta_t}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{\partial n_{t+s}}{\partial\zeta_t} &= \frac{\partial n_{t+s}}{\partial y_{t+s}}\frac{\partial y_{t+s}}{\partial\zeta_t} \\
&= \frac{1}{1-\alpha}\frac{\partial y_{t+s}}{\partial\zeta_t} \\
&= \frac{\tau_m\varphi_{y_0}+1}{1-\alpha}\sum_{q=0}^{s}\frac{\partial\widehat{y}_{t+s-q}}{\partial\zeta_t}+\frac{\tau_m\varphi_{y_1}-1}{1-\alpha}\sum_{q=0}^{s}\frac{\partial\widehat{y}_{t+s-q-1}}{\partial\zeta_t} \\
&\quad +\frac{\tau_m\varphi_\pi}{1-\alpha}\sum_{q=0}^{s}\frac{\partial\pi_{t+s-q}}{\partial\zeta_t}+\frac{\tau_m}{1-\alpha}\sum_{q=0}^{s}\phi_\zeta^q
\end{aligned}$$

for $s=1,2,3,...$, where $\tau_m\equiv\frac{\lambda}{(\sigma+\lambda)+\frac{\varphi+\alpha}{1-\alpha}}$.

B.7.6 Impulse Response of Bond Rate of Return with respect to Monetary Policy Shock

We have $\widetilde{i}_t = \phi_y E_t\{\Delta y_{t+1}\} - \phi_m E_t\{\Delta m_{t+1}\} + \phi_\pi E_t\{\Delta \pi_{t+1}\} + \varepsilon_{i,t}$. Using the fact that $\widehat{y}_t = y_t - y_t^n$,

$$\begin{aligned}
\widetilde{i}_t &= \phi_y E_t\{\Delta y_{t+1}\} - \phi_m E_t\{\Delta m_{t+1}\} + \phi_\pi E_t\{\Delta \pi_{t+1}\} + \varepsilon_{i,t} \\
&= \phi_y \left[E_t\{\Delta \widehat{y}_{t+1}\} + E_t\{\Delta y_{t+1}^n\}\right] - \phi_m E_t\{\Delta m_{t+1}\} + \phi_\pi E_t\{\Delta \pi_{t+1}\} + \varepsilon_{i,t}.
\end{aligned}$$

Using

$$\Delta y_t^n = \tau_a \Delta a_t + \tau_m \Delta m_t$$

where

$$\begin{aligned}
\tau_a &\equiv \frac{\frac{1+\varphi}{1-\alpha}}{(\sigma+\lambda) + \frac{\varphi+\alpha}{1-\alpha}} > 0 \\
\tau_m &\equiv \frac{\lambda}{(\sigma+\lambda) + \frac{\varphi+\alpha}{1-\alpha}}.
\end{aligned}$$

the above equation converts to

$$\widetilde{i}_t = \phi_y \left[E_t\{\Delta \widehat{y}_{t+1}\} + (\tau_a E_t\{\Delta a_{t+1}\} + \tau_m E_t\{\Delta m_{t+1}\})\right] - \phi_m E_t\{\Delta m_{t+1}\} + \phi_\pi E_t\{\Delta \pi_{t+1}\} + \varepsilon_{i,t}$$

rearrange the terms

$$\widetilde{i}_t = \phi_y E_t\{\Delta \widehat{y}_{t+1}\} + (\phi_y \tau_m - \phi_m)\, E_t\{\Delta m_{t+1}\} + \phi_\pi E_t\{\Delta \pi_{t+1}\} + \varepsilon_{i,t} + \phi_y \tau_a E_t\{\Delta a_{t+1}\}$$

using central bank's money supply function, $\Delta m_t = \varphi_{y_0}\widehat{y}_t + \varphi_{y_1}\widehat{y}_{t-1} + \varphi_\pi \pi_t + \zeta_t$ with $\zeta_t = \phi_\zeta \zeta_{t-1} + \varepsilon_{\zeta,t}$ we can have

$$
\begin{aligned}
\widetilde{i}_t &= \phi_y E_t\{\Delta \widehat{y}_{t+1}\} + (\phi_y \tau_m - \phi_m)\, E_t\{\varphi_{y_0}\widehat{y}_{t+1} + \varphi_{y_1}\widehat{y}_t + \varphi_\pi \pi_{t+1} + \zeta_{t+1}\} \\
&\quad + \phi_\pi E_t\{\Delta \pi_{t+1}\} + \varepsilon_{i,t} + \phi_y \tau_a E_t\{\Delta a_{t+1}\} \\
\widetilde{i}_t &= \phi_y E_t\{\widehat{y}_{t+1}\} - \phi_y \widehat{y}_t \\
&\quad + (\phi_y \tau_m - \phi_m)\, \varphi_{y_0} E_t\{\widehat{y}_{t+1}\} + (\phi_y \tau_m - \phi_m)\, \varphi_{y_1}\widehat{y}_t + (\phi_y \tau_m - \phi_m)\, \varphi_\pi E_t\{\pi_{t+1}\} \\
&\quad + (\phi_y \tau_m - \phi_m)\, E_t\{\zeta_{t+1}\} + \phi_\pi E_t\{\pi_{t+1}\} - \phi_\pi \pi_t + \varepsilon_{i,t} + \phi_y \tau_a E_t\{\Delta a_{t+1}\} \\
\widetilde{i}_t &= \left[(\phi_y \tau_m - \phi_m)\, \varphi_{y_0} + \phi_y\right] E_t\{\widehat{y}_{t+1}\} + \left[(\phi_y \tau_m - \phi_m)\, \varphi_{y_1} - \phi_y\right] \widehat{y}_t \\
&\quad + \left[(\phi_y \tau_m - \phi_m)\, \varphi_\pi + \phi_\pi\right] E_t\{\pi_{t+1}\} - \phi_\pi \pi_t \\
&\quad + \varepsilon_{i,t} + \phi_y \tau_a E_t\{\Delta a_{t+1}\} + (\phi_y \tau_m - \phi_m)\, \phi_\zeta \zeta_t
\end{aligned}
$$

Further, we are going to derive the analytical form of $E_t\{\Delta y_{t+1}\}$ and $E_t\{\Delta \pi_{t+1}\}$ by using

$$
\begin{aligned}
\widehat{y}_t &= \frac{B}{A} E_t\left\{\widehat{y}_{t+1}\right\} + \frac{C}{A} E_t\left\{\pi_{t+1}\right\} + \frac{1}{A} S_t \\
\pi_t &= \beta E_t\left\{\pi_{t+1}\right\} + \kappa \widehat{y}_t
\end{aligned}
$$

where

$$
S_t = \left(\tau_m - \frac{\lambda - \phi_m}{\sigma + \lambda - \phi_y}\right) \phi_\zeta \zeta_t + \tau_a(\phi_a - 1) a_t - \frac{1}{\sigma + \lambda - \phi_y} \varepsilon_{i,t}
$$

Solving for $E_t\left\{\widehat{y}_{t+1}\right\}$ and $E_t\left\{\pi_{t+1}\right\}$ we can get

$$
\begin{aligned}
E_t\left\{\pi_{t+1}\right\} &= \frac{1}{\beta}\left(\pi_t - \kappa \widehat{y}_t\right) \\
E_t\left\{\widehat{y}_{t+1}\right\} &= \left(\frac{\kappa}{\beta}\frac{C}{B} + \frac{A}{B}\right) \widehat{y}_t - \frac{1}{\beta}\frac{C}{B}\pi_t - \frac{1}{B} S_t
\end{aligned}
$$

therefore

$$
\begin{aligned}
\widetilde{i}_t &= \left[\left(\phi_y\tau_m-\phi_m\right)\varphi_{y_0}+\phi_y\right]E_t\{\widehat{y}_{t+1}\}+\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_{y_1}-\phi_y\right]\widehat{y}_t \\
&\quad+\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_\pi+\phi_\pi\right]E_t\{\pi_{t+1}\}-\phi_\pi\pi_t \\
&\quad+\varepsilon_{i,t}+\phi_y\tau_aE_t\{\Delta a_{t+1}\}+\left(\phi_y\tau_m-\phi_m\right)\phi_\zeta\zeta_t \\
&= \left[\left(\phi_y\tau_m-\phi_m\right)\varphi_{y_0}+\phi_y\right]\left[\left(\frac{\kappa}{\beta}\frac{C}{B}+\frac{A}{B}\right)\widehat{y}_t-\frac{1}{\beta}\frac{C}{B}\pi_t-\frac{1}{B}S_t\right] \\
&\quad+\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_{y_1}-\phi_y\right]\widehat{y}_t \\
&\quad+\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_\pi+\phi_\pi\right]\frac{1}{\beta}\left(\pi_t-\kappa\widehat{y}_t\right)-\phi_\pi\pi_t \\
&\quad+\varepsilon_{i,t}+\phi_y\tau_aE_t\{\Delta a_{t+1}\}+\left(\phi_y\tau_m-\phi_m\right)\phi_\zeta\zeta_t \\
&= \{\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_{y_0}+\phi_y\right]\left(\frac{\kappa}{\beta}\frac{C}{B}+\frac{A}{B}\right)+\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_{y_1}-\phi_y\right] \\
&\quad-\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_\pi+\phi_\pi\right]\frac{\kappa}{\beta}\}\widehat{y}_t \\
&\quad+\left(\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_\pi+\phi_\pi\right]\frac{1}{\beta}-\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_{y_0}+\phi_y\right]\frac{1}{\beta}\frac{C}{B}-\phi_\pi\right)\pi_t \\
&\quad-\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_{y_0}+\phi_y\right]\frac{1}{B}S_t+\varepsilon_{i,t}+\phi_y\tau_aE_t\{\Delta a_{t+1}\}+\left(\phi_y\tau_m-\phi_m\right)\phi_\zeta\zeta_t
\end{aligned}
$$

By turning off technology shock and bond rate shock, we can obtain

$$
\begin{aligned}
\widetilde{i}_t &= \{\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_{y_0}+\phi_y\right]\left(\frac{\kappa}{\beta}\frac{C}{B}+\frac{A}{B}\right)+\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_{y_1}-\phi_y\right] \\
&\quad-\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_\pi+\phi_\pi\right]\frac{\kappa}{\beta}\}\widehat{y}_t \\
&\quad+\left(\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_\pi+\phi_\pi\right]\frac{1}{\beta}-\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_{y_0}+\phi_y\right]\frac{1}{\beta}\frac{C}{B}-\phi_\pi\right)\pi_t \\
&\quad+\left(\left(\phi_y\tau_m-\phi_m\right)\phi_\zeta-\left[\left(\phi_y\tau_m-\phi_m\right)\varphi_{y_0}+\phi_y\right]\frac{1}{B}\left(\tau_m-\frac{\lambda-\phi_m}{\sigma+\lambda-\phi_y}\right)\phi_\zeta\right)\zeta_t
\end{aligned}
$$

Take derivative of $\widetilde{i}_t$ with respect to ζ_t

$$
\begin{aligned}
\frac{\partial \widetilde{i}_t}{\partial \zeta_t} &= \{[(\phi_y \tau_m - \phi_m)\varphi_{y_0} + \phi_y]\left(\frac{\kappa}{\beta}\frac{C}{B} + \frac{A}{B}\right) + [(\phi_y \tau_m - \phi_m)\varphi_{y_1} - \phi_y] \\
&\quad - [(\phi_y \tau_m - \phi_m)\varphi_\pi + \phi_\pi]\frac{\kappa}{\beta}\}\frac{\partial \widehat{y}_t}{\partial \zeta_t} \\
&\quad + \left([(\phi_y \tau_m - \phi_m)\varphi_\pi + \phi_\pi]\frac{1}{\beta} - [(\phi_y \tau_m - \phi_m)\varphi_{y_0} + \phi_y]\frac{1}{\beta}\frac{C}{B} - \phi_\pi\right)\frac{\partial \pi_t}{\partial \zeta_t} \\
&\quad + \left((\phi_y \tau_m - \phi_m)\phi_\zeta - [(\phi_y \tau_m - \phi_m)\varphi_{y_0} + \phi_y]\frac{1}{B}\left(\tau_m - \frac{\lambda - \phi_m}{\sigma + \lambda - \phi_y}\right)\phi_\zeta\right)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\frac{\partial \widetilde{i}_{t+s}}{\partial \zeta_t} &= \frac{\partial \widetilde{i}_{t+s}}{\partial \widehat{y}_{t+s}}\frac{\partial \widehat{y}_{t+s}}{\partial \zeta_t} + \frac{\partial \widetilde{i}_{t+s}}{\partial \pi_{t+s}}\frac{\partial \pi_{t+s}}{\partial \zeta_t} \\
&= \{[(\phi_y \tau_m - \phi_m)\varphi_{y_0} + \phi_y]\left(\frac{\kappa}{\beta}\frac{C}{B} + \frac{A}{B}\right) + [(\phi_y \tau_m - \phi_m)\varphi_{y_1} - \phi_y] \\
&\quad - [(\phi_y \tau_m - \phi_m)\varphi_\pi + \phi_\pi]\frac{\kappa}{\beta}\}\frac{\partial \widehat{y}_{t+s}}{\partial \zeta_t} \\
&\quad + \left([(\phi_y \tau_m - \phi_m)\varphi_\pi + \phi_\pi]\frac{1}{\beta} - [(\phi_y \tau_m - \phi_m)\varphi_{y_0} + \phi_y]\frac{1}{\beta}\frac{C}{B} - \phi_\pi\right)\frac{\partial \pi_{t+s}}{\partial \zeta_t}
\end{aligned}
$$

for $s = 1, 2, 3, ...$